

Conjunction and Negation of Natural Concepts: A Quantum-theoretic Modeling

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Abstract

We perform two experiments with the aim to investigate the effects of negation on the combination of natural concepts. In the first experiment, we test the membership weights of a list of exemplars with respect to two concepts, e.g., *Fruits* and *Vegetables*, and their conjunction *Fruits And Vegetables*. In the second experiment, we test the membership weights of the same list of exemplars with respect to the same two concepts, but negating the second, e.g., *Fruits* and *Not Vegetables*, and again their conjunction *Fruits And Not Vegetables*. The collected data confirm existing results on conceptual combination, namely, they show dramatic deviations from the predictions of classical (fuzzy set) logic and probability theory. More precisely, they exhibit conceptual vagueness, gradeness of membership, overextension and double overextension of membership weights with respect to the given conjunctions. Then, we show that the quantum probability model in Fock space recently elaborated to model Hampton's data on concept conjunction (Hampton, 1988a) and disjunction (Hampton, 1988b) faithfully accords with the collected data. Our quantum-theoretic modeling enables to describe these non-classical effects in terms of genuine quantum effects, namely 'contextuality', 'superposition', 'interference' and 'emergence'. The obtained results confirm and strenghten the analysis in Aerts (2009a) and Sozzo (2014) on the identification of quantum aspects in experiments on conceptual vagueness. Our results can be inserted within the general research on the identification of quantum structures in cognitive and decision processes.

1 Introduction

In the last years there has been a renewed interest in the formulation of a unified psychological theory for representing and structuring concepts. Indeed, traditional approaches to concept theory, mainly, 'prototype theory' (Rosch, 1973; Rosch, 1977; Rosch, 1983), 'exemplar theory' (Nosofsky, 1988; Nosofsky, 1992) and 'theory theory' (Murphy & Medin, 1985; Rumelhart & Norman, 1988) are still facing a crucial difficulty, namely, 'the problem of how modeling the combination of two or more natural concepts starting from the modeling of the component ones'. This 'combination problem' has been revealed by several cognitive experiments in the last thirty years. More precisely:

(i) The 'Guppy effect' in concept conjunction, also known as the 'Pet-Fish problem' (Osherson & Smith, 1981). If one measures the typicality of specific exemplars with respect to the concepts *Pet* and *Fish* and their conjunction *Pet-Fish*, then one experimentally finds that an exemplar such as *Guppy* is a very typical example of *Pet-Fish*, while it is neither a very typical example of *Pet* nor of *Fish*.

(ii) The deviation from classical (fuzzy) set-theoretic membership weights of exemplars with respect to pairs of concepts and their conjunction or disjunction (Hampton, 1988a,b). If one measures the mem-

bership weight of an exemplar with respect to a pair of concepts and their conjunction (disjunction), then one experimentally finds that the membership weight of the exemplar with respect to the conjunction (disjunction) is greater (less) than the membership weight of the exemplar with respect to at least one of the component concepts.

(iii) The so-called ‘borderline contradictions’ (Alxatib & Pelletier, 2011; Bonini, Osherson, Viale & Williamson, 1999). Roughly speaking, a borderline contradiction is a sentence of the form $P(x) \wedge \neg P(x)$, for a vague predicate P and a borderline case x , e.g., the sentence “Mark is rich and Mark is not rich”.

If one accepts that concepts are ‘graded’, or ‘fuzzy’, notions (Osherson & Smith, 1982; Zadeh, 1965, 1982), as empirical evidence seem to confirm, then one cannot represent the membership weights and typicalities expressing such gradeness in a classical (fuzzy) set-theoretic model, where conceptual conjunctions are represented logical conjunctions and conceptual disjunctions are represented by logical disjunctions. These difficulties affect both ‘extensional’ membership-based (Rips, 1995; Zadeh, 1982) and ‘intensional’ attribute-based (Hampton, 1988b, 1997; Minsky, 1975). This combination problem is considered so serious that many authors maintain that not much progress is possible in the field if no light is shed on this problem (Fodor, 1994; Hampton, 1997; Kamp & Partee, 1995; Komatsu, 1992; Rips, 1995). However no mechanism and/or procedure has as yet been identified that gives rise to a satisfactory description or explanation of the effects appearing when concepts combine.

Very similar effects and deviations from the predictions of traditional approaches have meanwhile been experienced in other domains of cognitive science, specifically, in behavioural economics (Ellsberg, 1961; Machina, 2009) and decision theory (Tversky & Kahneman, 1983; Tversky & Shafir, 1992). These and other difficulties have led various scholars to look for alternative approaches which could provide a more satisfactory picture of ‘what occurs in human thought in a cognitive or decision process’. Among the possible alternatives, a major candidate is what has been called ‘quantum cognition’ and it rests the application of the mathematical formalism of quantum theory in cognitive and social domains (see. e.g., Aerts, 2009a,b; Aerts, Broekaert, Gabora & Sozzo, 2013; Aerts & Czachor, 2004; Aerts & Gabora, 2005a,b; Aerts, Gabora & Sozzo, 2013; Aerts & Sozzo, 2011, 2013; Aerts, Sozzo & Tapia, 2014; Busemeyer & Bruza, 2012; Busemeyer, Pothos, Franco & Trueblood, 2011; Haven & Khrennikov, 2013; Khrennikov, 2010; Pothos & Busemeyer, 2009, 2013; van Rijsbergen, 2004; Wang, Busemeyer, Atmanspacher & Pothos, 2013).

In this paper, we mainly deal with the quantum-theoretic approach to cognitive science elaborated in Brussels. This approach was motivated by a two decade research on the foundations of quantum theory (Aerts, 1999), the origins of quantum probability (Aerts, 1986; Pitowsky, 1989) and the identification of typically quantum aspects in the macroscopic world (Aerts & Aerts, 1995; Aerts, Aerts, Broekaert & Gabora, 2000). A *SCoP formalism* was worked out within the Brussels approach which relies on the interpretation of a concept as an ‘entity in a specific state changing under the influence of a context’ rather than as a ‘container of instantiations’ (Aerts & Gabora, 2005a,b), and allowed the authors to provide a quantum representation of the guppy effect (Aerts & Gabora, 2005a,b). Successively, the mathematical formalism of quantum theory was employed to model the overextension and underextension of membership weights measured by Hampton (1988a,b). More specifically, the overextension for conjunctions of concepts measured by Hampton (1988a) was described as an effect of quantum interference, superposition and emergence (Aerts, 2009a; Aerts, Gabora & Sozzo, 2013), which also play a primary role in the description of both overextension and underextension for disjunctions of concepts (Hampton, 1988b). Furthermore, a specific conceptual combination experimentally revealed the presence of another genuine quantum effect, namely, entanglement (Aerts, 2009a,b; Aerts, Broekaert, Gabora & Sozzo, 2012; Aerts, Gabora & Sozzo, 2012; Aerts & Sozzo, 2011). Finally, this quantum-theoretic framework was successfully applied to describe borderline vagueness (Sozzo, 2014).

More specifically, in the present paper we generalize Aerts (2009a)’s analysis of Hampton’s overextension for the conjunction of two concepts, extending it to conjunctions and negations. Negative concepts have

been typically considered as ‘singular concepts’, since they do not have a prototype. Indeed, it is, for example, easy to determine the membership of a concept such as *Not Fruit*, but it does not seem that such a determination involves similarity with some prototype of *Not Fruit*. This is why one is naturally led to derive the negation of a concept from (fuzzy set) logical operations on the positively defined concept. There has been very little research on how human beings interpret and combine negated concepts. In this respect, Hampton (1997) performed a set of experiments in which he considered both conjunctions of the form *Tools Which Are Also Weapons* and conjunctions of the form *Tools Which Are Not Weapons*. As expected, his seminal work confirmed overextension in both conjunctions, also showing a violation of Boolean classical logical rules for the negation. These results were the starting point for our research in this paper, whose content can be summarized as follows.

In Section 2 we describe the two experiments we performed. In the first experiment, we tested the membership weights of four different sets of exemplars with respect to four pairs (A, B) of concepts and their conjunction ‘ A and B ’. In the second experiment, we tested the membership weights of the same four sets of exemplars with respect to the same four pairs (A, B) of concepts, but negating the second concept, hence actually considering A , ‘not B ’ and the conjunction ‘ A and not B ’. We observe that, already at this level, several exemplars exhibited overextension with respect to both ‘ A and B ’ and ‘ A and not B ’, hence we get a first clue that a deviation from classical (fuzzy set) logic and probability theory is at play in our experiments. A complete analysis of the ‘non-classicality’ underlying the collected data is presented in Section 3 where we prove two theorems on the representability of a given set of experimental data in a classical Kolmogorovian probability space, thus extending the analysis in (Aerts, 2009a) to negated concepts. By applying these theorems, we show that a large part of our data cannot be modeled in a classical Kolmogorovian space. Moreover, we notice that the deviations from classicality are of two types: (i) overextension of membership weights with respect to both conjunctions ‘ A and B ’ and ‘ A and not B ’, (ii) deviation of the negation ‘not B ’ of the concept B from the classical logical negation. This non-classical behaviour led us to inquire into the possibility of representing our data in a quantum-mechanical framework. After a brief overview of the rules of a quantum-theoretic modeling in Section 4, we develop this modeling for the combinations ‘ A and B ’ and ‘ A and not B ’ in Section 5, thus extending the analysis in (Aerts, 2009a). Finally, we draw our conclusions in Section 6, where we:

- (i) prove that a large number of the collected data can be represented in our quantum-theoretic modeling in Fock space;
- (ii) describe the observed deviations from classicality as a consequence of genuine quantum effects, such as, ‘contextuality’, ‘interference’, ‘superposition’ and ‘emergence’;
- (iii) provide a further support to the explanatory hypothesis we have recently put forward for the effectiveness of a quantum approach in cognitive and decision processes. According to this hypothesis, human thought is the superposition of a ‘quantum emergent thought’ and a ‘quantum logical thought’, and that the quantum modeling approach applied in Fock space enables this approach to general human thought, consisting of a superposition of these two modes, to be modeled.

We observe, to conclude this section, that the results obtained in the present paper confirm those in (Aerts, 2009a) on conceptual conjunction/disjunction and in (Sozzo, 2014) on borderline vagueness. Hence they can be considered as a further theoretical support towards the identification of quantum structures in cognition.

2 Description of the experiment

Hampton identified in his experiments systematic deviations from classical set (fuzzy set) conjunctions and disjunctions (Hampton, 1988a,b). More explicitly, if the membership weight of an exemplar x with respect to the conjunction ‘ A and B ’ of two concepts A and B is higher than the membership weight of

x with respect to one concept (both concepts), we say that the membership weight of x is ‘overextended’ (‘double overextended’) with respect to the conjunction (by abuse of language, one usually says that x is overextended with respect to the conjunction). If the membership weight of an exemplar x with respect to the disjunction ‘ A or B ’ of two concepts A and B is less than the membership weight of x with respect to one concept, we say that the membership weight of x is ‘overextended’ with respect to the disjunction (by abuse of language, one usually says that x is overextended with respect to the disjunction). These were the non-classical effects detected by Hampton in the combination of two concepts. Similar effects were identified by the same author in his experiments on conjunction and negation of two concepts (Hampton, 1997). The analysis by Aerts (2009a) evidenced other deviations from classicality in Hampton’s experiments. In this section we show that very similar deviations from classicality can be observed in our experiment on human subjects. But we first need to describe the experiment.

In our experiment, we considered four pairs of natural concepts, namely (*Home Furnishing, Furniture*), (*Spices, Herbs*), (*Pets, Farmyard Animals*) and (*Fruits, Vegetables*). For each pair, we considered 24 exemplars and measured their membership with respect to these pairs of concepts and suitable conjunctions of these pairs. The membership was estimated by using a ‘7-point scale’. The tested subjects were asked to choose a number from the set $\{+3, +2, +1, 0, -1, -2, -3\}$, where the positive numbers $+1$, $+2$ and $+3$ meant that they considered ‘the exemplar to be a member of the concept’ – $+3$ indicated a strong membership, $+1$ a relatively weak membership. The negative numbers -1 , -2 and -3 meant that the subject considered ‘the exemplar to be a non-member of the concept’ – -3 indicated a strong non-membership, -1 a relatively weak non-membership.

For the conceptual pair (*Home Furnishing, Furniture*), we asked 80 subjects to estimate the membership of the first set of 24 exemplars with respect to the concepts *Home Furnishing, Furniture* and the negation *Not Furniture*. Then, we asked 40 subjects to estimate the membership of the same set of 24 exemplars with respect to the conjunctions *Home Furnishing And Furniture* and *Home Furnishing And Not Furniture*. Subsequently, we calculated the corresponding membership weights. The results are reported in Tables 1a and 1b .

For the conceptual pair (*Spices, Herbs*), we asked 80 subjects to estimate the membership of the second set of 24 exemplars with respect to the concepts *Spices, Herbs* and the negation *Not Herbs*. Then, we asked 40 subjects to estimate the membership of the same set of 24 exemplars with respect to the conjunctions *Spices And Herbs* and *Spices And Not Herbs*. Subsequently, we calculated the corresponding membership weights. The results are reported in Tables 2a and 2b.

For the conceptual pair (*Pets, Farmyard Animals*), we asked 80 subjects to estimate the membership of the third set of 24 exemplars with respect to the concepts *Pets, Farmyard Animals* and the negation *Not Farmyard Animals*. Then, we asked 40 subjects to estimate the membership of the same set of 24 exemplars with respect to the conjunctions *Pets And Farmyard Animals* and *Pets And Not Farmyard Animals*. Subsequently, we calculated the corresponding membership weights. The results are reported in Tables 3a and 3b.

For the conceptual pair (*Fruits, Vegetables*), we asked 80 subjects to estimate the membership of the fourth set of 24 exemplars with respect to the concepts *Fruits, Vegetables* and the negation *Not Vegetables*. Then, we asked 40 subjects to estimate the membership of the same set of 24 exemplars with respect to the conjunctions *Fruits And Vegetables* and *Fruits And Not Vegetables*. Subsequently, we calculated the corresponding membership weights. The results are reported in Tables 4a and 4b.

Pure inspection of Tables 1-4 reveals that several exemplars present overextension with respect to both conjunctions ‘ A and B ’ and ‘ A and not B ’. For example, the membership weight of *Chili Pepper* with respect to *Spices* is 0.975, with respect to *Herbs* is 0.53125, while its membership weight with respect to the conjunction *Spices And Herbs* is 0.8 (Table 2.a), thus giving rise to overextension. Also, if we consider the membership weights of *Goldfish* with respect to *Pets* and *Farmward Animals*, we get 0.925 and 0.16875,

respectively, while its membership weight with respect to *Pets And Farmyard Animals* is 0.425 (Table 3.a). Even stronger deviations in the combination *Fruits And Vegetables*. For example, the exemplar *Broccoli* scores 0.09375 with respect to *Fruits*, 1 with respect to *Vegetables*, and 0.5875 with respect to *Fruits And Vegetables*. A similar pattern is observed for *Parsley*, which scores 0.01875 with respect to *Fruits*, 0.78125 with respect to *Vegetables* and 0.45 with respect to *Fruits And Vegetables* (Tables 4.a).

Overextension is also present when one concept is negated, that is, in the combination ‘*A* and not *B*’. Indeed, the membership weights of *Shelves* with respect to *Home Furnishing*, *Not Furniture* and *Home Furnishing And Not Furniture* is 0.85, 0.125 and 0.3875, respectively (Table 1.b). Then, *Pepper* scores 0.99375 with respect to *Spices*, 0.58125 with respect to *Not Herbs*, and 0.9 with respect to *Spices and Not Herbs* (Table 2.b). Finally, *Doberman Guard Dog* scores 0.88125 and 0.26875 with respect to *Pets* and *Farmyard Animals*, respectively, while it scores 0.55 with respect to *Pets And Farmyard Animals* (Table 3b).

Double overextension is also present in various cases and for both conjunctions ‘*A* and *B*’ and ‘*A* and not *B*’. For example, the membership weight of *Olive* with respect to *Fruits And Vegetables* is 0.65, which is greater than both 0.53125 and 0.63125, i.e. the membership weights of *Olive* with respect to *Fruits* and *Vegetables*, respectively (Table 4.a). Furthermore, *Prize Bull* scores 0.13125 with respect to *Pets* and 0.2625 with respect to *Not Farmyard Animals*, but its membership weight with respect to *Pets And Not Farmyard Animals* is 0.275 (Table 3b).

Our preliminary analysis above already shows that manifest deviations from classicality occur in the experiment we performed. When we say ‘deviations from classicality’ we actually mean that the collected data behave in such a way that they cannot generally be modeled by using the usual connectives of classical fuzzy set logic for conceptual conjunctions, neither the rules of classical probability for their membership weights. In order to systematically identify such deviations from classicality we need however a characterization of the representability of these data in a classical probability space. This is the content of the next section.

3 Classical models for conjunctions and negations of two concepts

We derive in this section necessary and sufficient conditions for the classicality of experimental data coming from the membership weights of two concepts *A* and *B* with respect to the conceptual negation ‘not *B*’ and the conjunctions ‘*A* and *B*’ and ‘*A* and not *B*’. More explicitly, we first derive the constraints that should be satisfied by the membership weights $\mu_x(A)$, $\mu_x(B)$ and $\mu_x(A \text{ and } B)$ of the exemplar *x* with respect to the concepts *A*, *B* and ‘*A* and *B*’, respectively, in order to represent these data in a classical probability model satisfying the axioms of Kolmogorov. Then, we derive the constraints that should be satisfied by the membership weights $\mu_x(A)$, $\mu_x(\text{not } B)$ and $\mu_x(A \text{ and not } B)$ of the exemplar *x* with respect to the concepts *A*, *B*, ‘not *B*’ and ‘*A* and not *B*’, respectively, in order to represent these data in a classical Kolmogorovian probability model. We follow here mathematical procedures that are similar to those employed in Aerts (2009a) for the classicality of conceptual conjunctions and disjunctions. Let us start by clearly defining what we mean by the notion of ‘classical’, or ‘Kolmogorovian’, probability model.

Let us start by the definition of a σ -algebra over a set.

Definition 1. A σ -algebra over a set Ω is a non-empty collection $\sigma(\Omega)$ of subsets of Ω that is closed under complementation and countable unions of its members. It is a Boolean algebra, completed to include countably infinite operations.

Measure structures are the most general classical structures devised by mathematicians and physicists to structure weights. A Kolmogorovian probability measure is such a measure applied to statistical data. It is called ‘Kolmogorovian’, because Andrey Kolmogorov was the first to axiomatize probability theory in this way (Kolmogorov, 1933).

Definition 2. A measure P is a function defined on a σ -algebra $\sigma(\Omega)$ over a set Ω and taking values in the extended interval $[0, \infty]$ such that the following three conditions are satisfied:

- (i) the empty set has measure zero;
- (ii) if E_1, E_2, E_3, \dots is a countable sequence of pairwise disjoint sets in $\sigma(\Omega)$, the measure of the union of all the E_i is equal to the sum of the measures of each E_i (countable additivity, or σ -additivity);
- (iii) the triple $(\Omega, \sigma(\Omega), P)$ satisfying (i) and (ii) is then called a measure space, and the members of $\sigma(\Omega)$ are called measurable sets.

A Kolmogorovian probability measure is a measure with total measure one. A Kolmogorovian probability space $(\Omega, \sigma(\Omega), P)$ is a measure space $(\Omega, \sigma(\Omega), P)$ such that P is a Kolmogorovian probability. The three conditions expressed in a mathematical way are:

$$P(\emptyset) = 0 \quad P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i) \quad P(\Omega) = 1 \quad (1)$$

Let us now come to the possibility to represent a set of experimental data on two concepts and their conjunction in a classical Kolmogorovian probability model.

Definition 3. We say that the membership weights $\mu_x(A)$, $\mu_x(B)$ and $\mu_x(A \text{ and } B)$ of the exemplar x with respect to the pair of concepts A and B and their conjunction ‘ A and B ’, respectively, can be represented in a classical Kolmogorovian probability model if there exists a Kolmogorovian probability space $(\Omega, \sigma(\Omega), P)$ and events $E_A, E_B \in \sigma(\Omega)$ of the events algebra $\sigma(\Omega)$ such that

$$P(E_A) = \mu_x(A) \quad P(E_B) = \mu_x(B) \quad \text{and} \quad P(E_A \cap E_B) = \mu_x(A \text{ and } B) \quad (2)$$

We can prove a useful theorem on the representability of the membership weights with respect to two concepts and their conjunction.

Theorem 1. The membership weights $\mu_x(A)$, $\mu_x(B)$ and $\mu_x(A \text{ and } B)$ of the exemplar x with respect to concepts A and B and their conjunction ‘ A and B ’, respectively, can be represented in a classical Kolmogorovian probability model if and only if they satisfy the following inequalities:

$$0 \leq \mu_x(A \text{ and } B) \leq \mu_x(A) \leq 1 \quad (3)$$

$$0 \leq \mu_x(A \text{ and } B) \leq \mu_x(B) \leq 1 \quad (4)$$

$$\mu_x(A) + \mu_x(B) - \mu_x(A \text{ and } B) \leq 1 \quad (5)$$

Proof. If $\mu_x(A)$, $\mu_x(B)$ and $\mu_x(A \text{ and } B)$ can be represented in a classical probability model, then there exists a Kolmogorovian probability space $(\Omega, \sigma(\Omega), P)$ and events $E_A, E_B \in \sigma(\Omega)$ such that $P(E_A) = \mu_x(A)$, $P(E_B) = \mu_x(B)$ and $P(E_A \cap E_B) = \mu_x(A \text{ and } B)$. From the general properties of a Kolmogorovian probability space it follows that we have $0 \leq P(E_A \cap E_B) \leq P(E_A) \leq 1$ and $0 \leq P(E_A \cap E_B) \leq P(E_B) \leq 1$, which proves that inequalities (3) and (4) are satisfied. From the same general properties of a Kolmogorovian probability space it also follows that we have $P(E_A \cup E_B) = P(E_A) + P(E_B) - P(E_A \cap E_B)$, and since $P(E_A \cup E_B) \leq 1$ we also have $P(E_A) + P(E_B) - P(E_A \cap E_B) \leq 1$. This proves that inequality

(5) is satisfied. We have now proved that for the classical conjunction data $\mu_x(A), \mu_x(B)$ and $\mu_x(A \text{ and } B)$ the three inequalities are satisfied.

Now suppose that we have an exemplar x whose membership weights $\mu_x(A), \mu_x(B), \mu_x(A \text{ and } B)$ with respect to the concepts A and B and their conjunction ‘ A and B ’ are such that inequalities (3), (4) and (5) are satisfied. We prove that, as a consequence, $\mu_x(A), \mu_x(B)$ and $\mu_x(A \text{ and } B)$ can be represented in a Kolmogorovian probability model. To this end we explicitly construct a Kolmogorovian probability space that models these data. Consider the set $\Omega = \{1, 2, 3, 4\}$ and $\sigma(\Omega) = \mathcal{P}(\Omega)$, the set of all subsets of Ω . We define

$$P(\{1\}) = \mu_x(A \text{ and } B) \quad (6)$$

$$P(\{2\}) = \mu_x(A) - \mu_x(A \text{ and } B) \quad (7)$$

$$P(\{3\}) = \mu_x(B) - \mu_x(A \text{ and } B) \quad (8)$$

$$P(\{4\}) = 1 - \mu_x(A) - \mu_x(B) + \mu_x(A \text{ and } B) \quad (9)$$

and further for an arbitrary subset $S \subseteq \{1, 2, 3, 4\}$ we define

$$P(S) = \sum_{a \in S} P(\{a\}) \quad (10)$$

Let us prove that $P : \sigma(\Omega) \rightarrow [0, 1]$ is a probability measure. To this end we need to prove that $P(S) \in [0, 1]$ for an arbitrary subset $S \subseteq \Omega$, and that the ‘sum formula’ for a probability measure is satisfied to comply with (1). The sum formula for a probability measure is satisfied because of definition (10). What remains to be proved is that $P(S) \in [0, 1]$ for an arbitrary subset $S \subseteq \Omega$. $P(\{1\}), P(\{2\}), P(\{3\})$ and $P(\{4\})$ are contained in $[0, 1]$ as a direct consequence of inequalities (3), (4) and (5). Further, we have $P(\{1, 2\}) = \mu_x(A), P(\{1, 3\}) = \mu_x(B), P(\{3, 4\}) = 1 - \mu_x(A), P(\{2, 4\}) = 1 - \mu_x(B), P(\{2, 3, 4\}) = 1 - \mu_x(A \text{ and } B)$ and $P(\{1, 2, 3\}) = \mu_x(A) + \mu_x(B) - \mu_x(A \text{ and } B)$, and all these are contained in $[0, 1]$ as a consequence of inequalities (3), (4) and (5). Consider $P(\{2, 3\}) = \mu_x(A) + \mu_x(B) - 2\mu_x(A \text{ and } B)$. From inequality (5) it follows that $\mu_x(A) + \mu_x(B) - 2\mu_x(A \text{ and } B) \leq \mu_x(A) + \mu_x(B) - \mu_x(A \text{ and } B) \leq 1$. Further, we have, following inequalities (3) and (4), $\mu_x(A \text{ and } B) \leq \mu_x(A)$ and $\mu_x(A \text{ and } B) \leq \mu_x(B)$ and hence $2\mu_x(A \text{ and } B) \leq \mu_x(A) + \mu_x(B)$. From this it follows that $0 \leq \mu_x(A) + \mu_x(B) - 2\mu_x(A \text{ and } B)$. Hence we have proved that $P(\{2, 3\}) = \mu_x(A) + \mu_x(B) - 2\mu_x(A \text{ and } B) \in [0, 1]$. We have $P(\{1, 4\}) = 1 - \mu_x(A) - \mu_x(B) + 2\mu_x(A \text{ and } B) = 1 - P(\{2, 3\})$ and hence $P(\{1, 4\}) \in [0, 1]$. We have $P(\{1, 2, 4\}) = 1 - \mu_x(B) + \mu_x(A \text{ and } B) = 1 - P(\{3\}) \in [0, 1]$ and $P(\{1, 3, 4\}) = 1 - \mu_x(A) + \mu_x(A \text{ and } B) = 1 - P(\{2\}) \in [0, 1]$. The last subset to control is Ω itself. We have $P(\Omega) = P(\{1\}) + P(\{2\}) + P(\{3\}) + P(\{4\}) = 1$. We have verified all subsets $S \subseteq \Omega$, and hence proved that P is a probability measure. Since $P(\{1\}) = \mu_x(A \text{ and } B)$, $P(\{1, 2\}) = \mu_x(A)$ and $P(\{1, 3\}) = \mu_x(B)$, we have modeled the data $\mu_x(A), \mu_x(B)$ and $\mu_x(A \text{ and } B)$ by means of a Kolmogorovian probability space, and hence they are classical conjunction data. \square

Inequalities (3) and (4) hold if and only if the quantity $\Delta_{AB}(x) = \mu_x(A \text{ and } B) - \min(\mu_x(A), \mu_x(B)) \leq 0$. The quantity $\Delta_{AB}(x)$ is called the ‘conjunction minimum rule deviation’, since it expresses compatibility with the ‘minimum rule for the conjunction’ in fuzzy set theory. A situation where $\Delta_{AB}(x) > 0$ was called ‘overextension’ by Hampton (1988a). The quantity $k_{AB}(x) = 1 - \mu_x(A) - \mu_x(B) + \mu_x(A \text{ and } B)$ is called the ‘Kolmogorovian conjunction factor’. Its violation is due to a non-classicality that is different from the one entailing the violation $\Delta_{AB}(x)$ (Aerts, 2009a). Finally, let us introduce the quantity $\text{Doub}_{AB}(x) = \max(\mu_x(A), \mu_x(B)) - \mu_x(A \text{ and } B)$. A situation where $\text{Doub}_{AB}(x) > 0$ was called ‘double overextension’ by Hampton (1988a). The values of the parameters $\Delta_{AB}(x)$, $k_{AB}(x)$ and $\text{Doub}_{AB}(x)$ for our experiment are reported in Tables 1a, 2a, 3a and 4a.

Let us then come to the representability of a set of experimental data on a concept and its negation in a classical Kolmogorovian probability model.

Definition 4. We say that the membership weights $\mu_x(B)$ and $\mu_x(\text{not } B)$ of the exemplar x with respect to the concept B and its negation ‘not B ’, respectively, can be represented in a classical Kolmogorovian probability model if there exists a Kolmogorovian probability space $(\Omega, \sigma(\Omega), P)$ and an event $E_B \in \sigma(\Omega)$ of the events algebra $\sigma(\Omega)$ such that

$$P(E_B) = \mu_x(B) \quad P(\Omega \setminus E_B) = \mu_x(\text{not } B) \quad (11)$$

Analogously to the conjunction case, we can prove a useful theorem on the representability of the membership weights with respect to a positive concept A , a negated concept ‘not B ’ and their conjunction ‘ A and not B ’.

Theorem 2. The membership weights $\mu_x(A)$, $\mu_x(B)$, $\mu_x(\text{not } B)$ and $\mu_x(A \text{ and not } B)$ of the exemplar x with respect to the pair of concepts A , B , the negation ‘not B ’ and the conjunction ‘ A and not B ’, respectively, can be represented in a classical Kolmogorovian probability model if and only if they satisfy the following inequalities:

$$0 \leq \mu_x(A \text{ and not } B) \leq \mu_x(A) \leq 1 \quad (12)$$

$$0 \leq \mu_x(A \text{ and not } B) \leq \mu_x(\text{not } B) \leq 1 \quad (13)$$

$$\mu_x(A) + \mu_x(\text{not } B) - \mu_x(A \text{ and not } B) \leq 1 \quad (14)$$

$$1 - \mu_x(B) - \mu_x(\text{not } B) = 0 \quad (15)$$

Proof. If $\mu_x(A)$, $\mu_x(\text{not } B)$ and $\mu_x(A \text{ and not } B)$ can be represented in a classical probability model, then there exists a Kolmogorovian probability space $(\Omega, \sigma(\Omega), P)$ and events $E_A, E_B \in \sigma(\Omega)$ such that $P(E_A) = \mu_x(A)$, $P(\Omega \setminus E_B) = \mu_x(\text{not } B)$ and $P(E_A \cap (\Omega \setminus E_B)) = \mu_x(A \text{ and not } B)$. From the general properties of a Kolmogorovian probability space it follows that we have $0 \leq P(E_A \cap (\Omega \setminus E_B)) \leq P(E_A) \leq 1$ and $0 \leq P(E_A \cap (\Omega \setminus E_B)) \leq P(\Omega \setminus E_B) \leq 1$, which proves that inequalities (12) and (13) are satisfied. From the same general properties of a Kolmogorovian probability space it also follows that we have $P(E_A \cup (\Omega \setminus E_B)) = P(E_A) + P(\Omega \setminus E_B) - P(E_A \cap (\Omega \setminus E_B))$, and since $P(E_A \cup (\Omega \setminus E_B)) \leq 1$ we also have $P(E_A) + P(\Omega \setminus E_B) - P(E_A \cap (\Omega \setminus E_B)) \leq 1$. This proves that inequality (14) is satisfied. Finally, we have $P(E_B) + P(\Omega \setminus E_B) = P(\Omega) = 1$, in a Kolmogorovian probability space, which proves that inequality (15) is satisfied. We have now proved that for the classical conjunction data $\mu_x(A)$, $\mu_x(B)$, $\mu_x(\text{not } B)$ and $\mu_x(A \text{ and not } B)$ the three inequalities are satisfied.

Now suppose that we have an exemplar x whose membership weights $\mu_x(A)$, $\mu_x(\text{not } B)$, $\mu_x(A \text{ and not } B)$ with respect to the concepts A and ‘not B ’ and their conjunction ‘ A and not B ’ are such that inequalities (12), (13), (14) and (15) are satisfied. We prove that, as a consequence, $\mu_x(A)$, $\mu_x(\text{not } B)$ and $\mu_x(A \text{ and not } B)$ can be represented in a Kolmogorovian probability model. To this end we explicitly construct a Kolmogorovian probability space that models these data. Consider the set $\Omega = \{1, 2, 3, 4\}$ and $\sigma(\Omega) = \mathcal{P}(\Omega)$, the set of all subsets of Ω . We define

$$P(\{1\}) = \mu_x(A \text{ and not } B) \quad (16)$$

$$P(\{2\}) = \mu_x(A) - \mu_x(A \text{ and not } B) \quad (17)$$

$$P(\{3\}) = \mu_x(\text{not } B) - \mu_x(A \text{ and not } B) \quad (18)$$

$$P(\{4\}) = 1 - \mu_x(A) - \mu_x(\text{not } B) + \mu_x(A \text{ and not } B) \quad (19)$$

and further for an arbitrary subset $S \subseteq \{1, 2, 3, 4\}$ we define

$$P(S) = \sum_{a \in S} P(\{a\}) \quad (20)$$

Let us prove that $P : \sigma(\Omega) \rightarrow [0, 1]$ is a probability measure. To this end we need to prove that $P(S) \in [0, 1]$ for an arbitrary subset $S \subseteq \Omega$, and that the ‘sum formula’ for a probability measure is satisfied to comply with (1). The sum formula for a probability measure is satisfied because of definition (20). What remains to be proved is that $P(S) \in [0, 1]$ for an arbitrary subset $S \subseteq \Omega$. $P(\{1\})$, $P(\{2\})$, $P(\{3\})$ and $P(\{4\})$ are contained in $[0, 1]$ as a direct consequence of inequalities (12), (13) and (14). Further, we have $P(\{1, 2\}) = \mu_x(A)$, $P(\{1, 3\}) = \mu_x(\text{not } B)$, $P(\{3, 4\}) = 1 - \mu_x(A)$, $P(\{2, 4\}) = 1 - \mu_x(\text{not } B) = \mu_x(B)$, because of Equation (15), $P(\{2, 3, 4\}) = 1 - \mu_x(A \text{ and not } B)$ and $P(\{1, 2, 3\}) = \mu_x(A) + \mu_x(\text{not } B) - \mu_x(A \text{ and not } B)$, and all these are contained in $[0, 1]$ as a consequence of inequalities (12), (13) and (14). Consider $P(\{2, 3\}) = \mu_x(A) + \mu_x(B) - 2\mu_x(A \text{ and not } B)$. From inequality (14) it follows that $\mu_x(A) + \mu_x(\text{not } B) - 2\mu_x(A \text{ and not } B) \leq \mu_x(A) + \mu_x(\text{not } B) - \mu_x(A \text{ and not } B) \leq 1$. Further, we have, following inequalities (12) and (13), $\mu_x(A \text{ and not } B) \leq \mu_x(A)$ and $\mu_x(A \text{ and not } B) \leq \mu_x(\text{not } B)$ and hence $2\mu_x(A \text{ and not } B) \leq \mu_x(A) + \mu_x(\text{not } B)$. From this it follows that $0 \leq \mu_x(A) + \mu_x(\text{not } B) - 2\mu_x(A \text{ and not } B)$. Hence we have proved that $P(\{2, 3\}) = \mu_x(A) + \mu_x(\text{not } B) - 2\mu_x(A \text{ and not } B) \in [0, 1]$. We have $P(\{1, 4\}) = 1 - \mu_x(A) - \mu_x(\text{not } B) + 2\mu_x(A \text{ and not } B) = 1 - P(\{2, 3\})$ and hence $P(\{1, 4\}) \in [0, 1]$. We have $P(\{1, 2, 4\}) = 1 - \mu_x(\text{not } B) + \mu_x(A \text{ and not } B) = 1 - P(\{3\}) \in [0, 1]$ and $P(\{1, 3, 4\}) = 1 - \mu_x(A) + \mu_x(A \text{ and not } B) = 1 - P(\{2\}) \in [0, 1]$. The last subset to control is Ω itself. We have $P(\Omega) = P(\{1\}) + P(\{2\}) + P(\{3\}) + P(\{4\}) = 1$. We have verified all subsets $S \subseteq \Omega$, and hence proved that P is a probability measure. Since $P(\{1\}) = \mu_x(A \text{ and not } B)$, $P(\{1, 2\}) = \mu_x(A)$, $P(\{1, 3\}) = \mu_x(\text{not } B)$, $P(\{2, 4\}) = \mu_x(B)$ and $P(\Omega \setminus \{2, 4\}) = \mu_x(\text{not } B)$, we have modeled the data $\mu_x(A)$, $\mu_x(B)$ and $\mu_x(A \text{ and } B)$ by means of a Kolmogorovian probability space, and hence they are classical conjunction data. \square

Inequalities (12) and (13) hold if and only if the conjunction minimum rule deviation $\Delta_{AB'}(x) = \mu_x(A \text{ and not } B) - \min(\mu_x(A), \mu_x(\text{not } B)) \leq 0$. A situation where $\Delta_{AB'}(x) > 0$ entails that ‘overextension’ is present. The Kolmogorovian conjunction factor $k_{AB'}(x) = 1 - \mu_x(A) - \mu_x(\text{not } B) + \mu_x(A \text{ and not } B) \leq 0$ and the quantity $l_{BB'}(x) = 1 - \mu_x(B) - \mu_x(\text{not } B) = 0$ complete the classicality of the conjunction ‘A and not B’. Then, let us introduce the quantity $\text{Doub}_{AB'}(x) = \max(\mu_x(A), \mu_x(\text{not } B)) - \mu_x(A \text{ and not } B)$. A situation where $\text{Doub}_{AB'}(x) > 0$ is a situation of ‘double overextension’. The quantity $l_{BB'}(x) = 1 - \mu_x(B) - \mu_x(\text{not } B)$ in Equation (15) is a new parameter that must be introduced to represent the negation of a concept in terms of a classical set-theoretic complementation, namely,. A situation where $l_{BB'}(x) \neq 0$ produces a type of deviation from classicality and it is due to conceptual negation. The values of the parameters $\Delta_{AB}(x)$, $k_{AB}(x)$, $\text{Doub}_{AB}(x)$ and $l_{BB'}(x)$ for our experiment are reported in Tables 1b, 2b, 3b and 4b.

Let us now come back to our experiments. Theorems 1 and 2 are manifestly violated by several exemplars with respect to both conjunctions ‘A and B’ and ‘A and not B’. It is however interesting to observe that the conditions $k_{AB}(x) > 0$ and $k_{AB'}(x) > 0$ are never violated, hence the deviations from a classical probability model in our experimental data are all due to overextension in the conjunctions ‘A and B’ and ‘A and not B’ and to a violation of $l_{BB'} = 0$ in the negation not B. For example, the item *Prize Bull* has $\Delta_{AB}(x) = 0.29375 > 0$ with respect to *Pets And Farmyard Animals*, hence it is strongly overextended with respect to *Pets And Farmyard Animals*, and it is even double overextended with $\text{Doub}_{AB'}(x) = -0.0125 < 0$ with respect to *Pets and Not Farmyard Animals*. The already mentioned *Broccoli* and *Parsley* are such that their $\Delta_{AB}(x)$ s are equal to 0.43125 and 0.49375, respectively. The exemplar *Chili Pepper* has $\Delta_{AB'}(x) = 0.3375$, while the exemplar *Broccoli* has $\Delta_{AB'}(x) = 0.31875$, both with respect to *Fruits And Not Vegetables*, hence they are both highly overextended.

It is finally interesting to observe that evident deviations from classicality are also due to conceptual negation. Let us consider some cases. The exemplar *Rug* has $l_{BB'}(x) = -0.18125$ with respect to the concept *Furniture* and its negation *Not Furniture*, while *Wall Mirror* has $l_{BB'}(x) = -0.20625$ with respect to the same concept and negation. Other relevant examples are *Sugar* and *Chives* with $l_{BB'}(x) = -0.1125$ and $l_{BB'}(x) = -0.14375$, respectively, with respect to *Herbs* and *Not Herbs*, and *Collie Dog* with $l_{BB'}(x) = -0.1188$ with respect to *Farmyard Animals* and *Not Farmyard Animals*.

The results obtained in this section point to a systematic deviation of our experimental data from the rules of classical (fuzzy set) logic and probability theory. It is then worth to investigate whether the ‘non-classicalities’ identified here are of a quantum-type, and hence they can be described within the mathematical formalism of quantum theory. To this end we need to preliminary summarize the essentials of the quantum mathematics that is needed to employ this quantum formalism for modeling purposes.

4 Fundamentals of a quantum-theoretic modeling

We illustrate in this section how the mathematical formalism of quantum theory can be applied to model situations outside the microscopic quantum world, more specifically, in the representation of concepts and their combinations. We avoid in our presentation superfluous technicalities, but aim to be synthetic and rigorous at the same time.

When the quantum mechanical formalism is applied for modeling purposes, each considered entity – in our case a concept – is associated with a complex Hilbert space \mathcal{H} , that is, a vector space over the field \mathbb{C} of complex numbers, equipped with an inner product $\langle \cdot | \cdot \rangle$ that maps two vectors $\langle A |$ and $| B \rangle$ onto a complex number $\langle A | B \rangle$. We denote vectors by using the bra-ket notation introduced by Paul Adrien Dirac, one of the pioneers of quantum theory (Dirac, 1958). Vectors can be ‘kets’, denoted by $| A \rangle$, $| B \rangle$, or ‘bras’, denoted by $\langle A |$, $\langle B |$. The inner product between the ket vectors $| A \rangle$ and $| B \rangle$, or the bra-vectors $\langle A |$ and $\langle B |$, is realized by juxtaposing the bra vector $\langle A |$ and the ket vector $| B \rangle$, and $\langle A | B \rangle$ is also called a ‘bra-ket’, and it satisfies the following properties:

- (i) $\langle A | A \rangle \geq 0$;
- (ii) $\langle A | B \rangle = \langle B | A \rangle^*$, where $\langle B | A \rangle^*$ is the complex conjugate of $\langle A | B \rangle$;
- (iii) $\langle A | (z| B \rangle + t| C \rangle) = z\langle A | B \rangle + t\langle A | C \rangle$, for $z, t \in \mathbb{C}$, where the sum vector $z| B \rangle + t| C \rangle$ is called a ‘superposition’ of vectors $| B \rangle$ and $| C \rangle$ in the quantum jargon.

From (ii) and (iii) follows that inner product $\langle \cdot | \cdot \rangle$ is linear in the ket and anti-linear in the bra, i.e. $(z\langle A | + t\langle B |)| C \rangle = z^*\langle A | C \rangle + t^*\langle B | C \rangle$.

We recall that the ‘absolute value’ of a complex number is defined as the square root of the product of this complex number times its complex conjugate, that is, $| z | = \sqrt{z^* z}$. Moreover, a complex number z can either be decomposed into its cartesian form $z = x + iy$, or into its goniometric form $z = | z | e^{i\theta} = | z | (\cos \theta + i \sin \theta)$. As a consequence, we have $|\langle A | B \rangle| = \sqrt{\langle A | B \rangle \langle B | A \rangle}$. We define the ‘length’ of a ket (bra) vector $| A \rangle$ ($\langle A |$) as $|| | A \rangle || = || \langle A | || = \sqrt{\langle A | A \rangle}$. A vector of unitary length is called a ‘unit vector’. We say that the ket vectors $| A \rangle$ and $| B \rangle$ are ‘orthogonal’ and write $| A \rangle \perp | B \rangle$ if $\langle A | B \rangle = 0$.

We have now introduced the necessary mathematics to state the first modeling rule of quantum theory, as follows.

First quantum modeling rule: A state A of an entity – in our case a concept – modeled by quantum theory is represented by a ket vector $| A \rangle$ with length 1, that is $\langle A | A \rangle = 1$.

An orthogonal projection M is a linear operator on the Hilbert space, that is, a mapping $M : \mathcal{H} \rightarrow \mathcal{H}$, $| A \rangle \mapsto M| A \rangle$ which is Hermitian and idempotent. The latter means that, for every $| A \rangle, | B \rangle \in \mathcal{H}$ and $z, t \in \mathbb{C}$, we have:

- (i) $M(z| A \rangle + t| B \rangle) = zM| A \rangle + tM| B \rangle$ (linearity);
- (ii) $\langle A | M| B \rangle = \langle B | M| A \rangle$ (hermiticity);
- (iii) $M \cdot M = M$ (idempotency).

The identity operator $\mathbb{1}$ maps each vector onto itself and is a trivial orthogonal projection. We say that two orthogonal projections M_k and M_l are orthogonal operators if each vector contained in $M_k(\mathcal{H})$ is orthogonal to each vector contained in $M_l(\mathcal{H})$, and we write $M_k \perp M_l$, in this case. The orthogonality of the projection operators M_k and M_l can also be expressed by $M_k M_l = 0$, where 0 is the null operator. A

set of orthogonal projection operators $\{M_k \mid k = 1, \dots, n\}$ is called a ‘spectral family’ if all projectors are mutually orthogonal, that is, $M_k \perp M_l$ for $k \neq l$, and their sum is the identity, that is, $\sum_{k=1}^n M_k = \mathbb{1}$.

The above definitions give us the necessary mathematics to state the second modeling rule of quantum theory, as follows.

Second quantum modeling rule: A measurable quantity Q of an entity – in our case a concept – modeled by quantum theory, and having a set of possible real values $\{q_1, \dots, q_n\}$ is represented by a spectral family $\{M_k \mid k = 1, \dots, n\}$ in the following way. If the entity – in our case a concept – is in a state represented by the vector $|A\rangle$, then the probability of obtaining the value q_k in a measurement of the measurable quantity Q is $\langle A|M_k|A\rangle = \|M_k|A\rangle\|^2$. This formula is called the ‘Born rule’ in the quantum jargon. Moreover, if the value q_k is actually obtained in the measurement, then the initial state is changed into a state represented by the vector

$$|A_k\rangle = \frac{M_k|A\rangle}{\|M_k|A\rangle\|} \quad (21)$$

This change of state is called ‘collapse’ in the quantum jargon.

The tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$ of two Hilbert spaces \mathcal{H}_A and \mathcal{H}_B is the Hilbert space generated by the set $\{|A_i\rangle \otimes |B_j\rangle\}$, where $|A_i\rangle$ and $|B_j\rangle$ are vectors of \mathcal{H}_A and \mathcal{H}_B , respectively, which means that a general vector of this tensor product is of the form $\sum_{ij} c_{ij} |A_i\rangle \otimes |B_j\rangle$. This gives us the necessary mathematics to introduce the third modeling rule.

Third quantum modeling rule: A state C of a compound entity – in our case a combined concept – is represented by a unit vector $|C\rangle$ of the tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$ of the two Hilbert spaces \mathcal{H}_A and \mathcal{H}_B containing the vectors that represent the states of the component entities – concepts.

The above means that we have $|C\rangle = \sum_{ij} c_{ij} |A_i\rangle \otimes |B_j\rangle$, where $|A_i\rangle$ and $|B_j\rangle$ are unit vectors of \mathcal{H}_A and \mathcal{H}_B , respectively, and $\sum_{ij} |c_{ij}|^2 = 1$. We say that the state C represented by $|C\rangle$ is a product state if it is of the form $|A\rangle \otimes |B\rangle$ for some $|A\rangle \in \mathcal{H}_A$ and $|B\rangle \in \mathcal{H}_B$. Otherwise, C is called an ‘entangled state’.

The Fock space is a specific type of Hilbert space, originally introduced in quantum field theory. For most states of a quantum field the number of identical quantum entities is not conserved but is a variable quantity. The Fock space copes with this situation in allowing its vectors to be superpositions of vectors pertaining to different sectors for fixed numbers of identical quantum entities. More explicitly, the k -th sector of a Fock space describes a fixed number of k identical quantum entities, and it is of the form $\mathcal{H} \otimes \dots \otimes \mathcal{H}$ of the tensor product of k identical Hilbert spaces \mathcal{H} . The Fock space \mathcal{F} itself is the direct sum of all these sectors, hence

$$\mathcal{F} = \bigoplus_{k=1}^j \bigotimes_{l=1}^k \mathcal{H} \quad (22)$$

For our modeling we have only used Fock space for the ‘two’ and ‘one quantum entity’ case, hence $\mathcal{F} = \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H})$. This is due to considering only combinations of two concepts. The sector \mathcal{H} is called the ‘sector 1’, while the sector $\mathcal{H} \otimes \mathcal{H}$ is called the ‘sector 2’. A unit vector $|F\rangle \in \mathcal{F}$ is then written as $|F\rangle = ne^{i\gamma}|C\rangle + me^{i\delta}(|A\rangle \otimes |B\rangle)$, where $|A\rangle, |B\rangle$ and $|C\rangle$ are unit vectors of \mathcal{H} , and such that $n^2 + m^2 = 1$. For combinations of j concepts, the general form of Fock space expressed in Equation (22) will have to be used.

This quantum-theoretic modeling can be generalized by allowing states to be represented by the so called ‘density operators’ and measurements to be represented by the so called ‘positive operator valued measures’. However, our representation above is sufficient for attaining the results in this paper and we will use it in the following sections.

5 A quantum model for the combination of two concepts

In this section, we put forward the quantum-theoretic framework that has been employed to model Hampton's (Hampton, 1988a,b) and Alxatib & Pelletier's (Alxatib & Pelletier, 2011), applying it to our experiment reported in Section 2. We show that this framework, once specified for the given conceptual combinations, i.e. 'A and B' and 'A and not B', enables a complete and successful modeling of those experimental data collected in Section 2.

Let us start from the disjunction 'A or B' of two concepts A and B. When the membership of the exemplar (item) x with respect to A is measured, we represent A by the unit vector $|A_d(x)\rangle$ of a Hilbert space \mathcal{H} , and describe the decision measurement of a subject estimating whether x is a member of A by means of a dichotomic observable represented by the orthogonal projection operator M . The probability $\mu_x(A)$ that x is chosen as a member of A, i.e. its membership weight, is given by the scalar product $\mu_x(A) = \langle A_d(x)|M|A_d(x)\rangle$. Let A and B be two concepts, represented by the unit vectors $|A_d(x)\rangle$ and $|B_d(x)\rangle$, respectively. To represent the concept 'A or B' we take the archetypical situation of the quantum double slit experiment, where $|A\rangle$ and $|B\rangle$ represent the states of a quantum particle in which only one slit is open, $\frac{1}{\sqrt{2}}(|A\rangle + |B\rangle)$ represents the state of the quantum particle in which both slits are open, and $\mu_x(A \text{ or } B)$ is the probability that the quantum particle is detected in a given region of a screen behind the slits. Thus, the concept 'A or B' is represented by the unit vector $\frac{1}{\sqrt{2}}(|A_d(x)\rangle + |B_d(x)\rangle)$, and $|A_d(x)\rangle$ and $|B_d(x)\rangle$ are chosen to be orthogonal, that is, $\langle A_d(x)|B_d(x)\rangle = 0$. The membership weights $\mu_x(A)$, $\mu_x(B)$ and $\mu_x(A \text{ or } B)$ of an exemplar x for the concepts A, B and 'A or B' are given by

$$\mu_x(A) = \langle A_d(x)|M|A_d(x)\rangle \quad (23)$$

$$\mu_x(B) = \langle B_d(x)|M|B_d(x)\rangle \quad (24)$$

$$\mu_x(A \text{ or } B) = \frac{1}{2}(\mu_x(A) + \mu_x(B)) + \Re\langle A_d(x)|M|B_d(x)\rangle \quad (25)$$

respectively, where $\Re\langle A_d(x)|M|B_d(x)\rangle$ is the real part of the complex number $\langle A_d(x)|M|B_d(x)\rangle$. The complex term $\Re\langle A_d(x)|M|B_d(x)\rangle$ is called 'interference term' in the quantum jargon, since it produces a deviation from the average $\frac{1}{2}(\mu_x(A) + \mu_x(B))$ which would have been observed in the quantum double slit experiment in absence of interference. We can see that, already at this stage, two genuine quantum effects, namely, superposition and interference, occur in the mechanism of combination of the concepts A and B.

The quantum-theoretic modeling presented above correctly describes a large part of data in Hampton (1988b), but it cannot cope with quite some cases – in fact most of all the cases that behave more classically than the ones that are easily modeled by quantum interference. The reason is that, if one wants to reproduce Hampton's data within a quantum mathematics model which fully exploits the analogy with the quantum double slit experiment, one has to include the situation in which two identical quantum particles are considered, both particles passes through the slits, and the probability that at least one particle is detected in the spot x is calculated. This probability is given by $\mu_x(A) + \mu_x(B) - \mu_x(A)\mu_x(B)$ (Aerts, 2009a). Quantum field theory in Fock space allows one to complete the model, as follows.

In quantum field theory, a quantum entity is described by a field which consists of superpositions of different configurations of many quantum particles (see Section 4). Thus, the quantum entity is associated with a Fock space \mathcal{F} which is the direct sum \oplus of different Hilbert spaces, each Hilbert space describing a defined number of quantum particles. In the simplest case, $\mathcal{F} = \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H})$, where \mathcal{H} is the Hilbert space of a single quantum particle (sector 1 of \mathcal{F}) and $\mathcal{H} \otimes \mathcal{H}$ is the (tensor product) Hilbert space of two identical quantum particles (sector 2 of \mathcal{F}).

Let us come back to our modeling for concept combinations. The normalized superposition $\frac{1}{\sqrt{2}}(|A_d(x)\rangle + |B_d(x)\rangle)$ represents the state of the new emergent concept 'A or B' in sector 1 of the Fock space \mathcal{F} . In sector 2 of \mathcal{F} , instead, the state of the concept 'A or B' is represented by the unit (product) vector $|A_d(x)\rangle \otimes$

$|B_d(x)\rangle$. To describe the decision measurement in this sector, we first suppose that the subject considers two identical copies of the exemplar x , pondering on the membership of the first copy of x with respect to A ‘and’ the membership of the second copy of x with respect to B . The probability of getting ‘yes’ in both cases is, by using quantum mechanical rules, $(\langle A_d(x)|\langle B_d(x)|)|M \otimes M|(|A_d(x)\rangle \otimes |B_d(x)\rangle)$. The probability of getting at least a positive answer is instead $1 - (\langle A_d(x)|\langle B_d(x)|)|(\mathbb{1} - M) \otimes (\mathbb{1} - M)|(|A_d(x)\rangle \otimes |B_d(x)\rangle)$. Hence, the membership weight of the exemplar x with respect to the concept ‘ A or B ’ coincides in sector 2 with the latter probability and can be written as $1 - (\langle A_d(x)|\langle B_d(x)|)|(\mathbb{1} - M) \otimes (\mathbb{1} - M)|(|A_d(x)\rangle \otimes |B_d(x)\rangle) = \mu_x(A) + \mu_x(B) - \mu_x(A)\mu_x(B) = (\langle A_d(x)|\langle B_d(x)|)|M \otimes \mathbb{1} + \mathbb{1} \otimes M - M \otimes M|(|A_d(x)\rangle \otimes |B_d(x)\rangle)$.

Coming to the Fock space $\mathcal{F} = \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H})$, the global initial state of the concepts is represented by the unit vector

$$|A \text{ or } B(x)\rangle = m_d(x)e^{i\lambda_d(x)}|A_d(x)\rangle \otimes |B_d(x)\rangle + n_d(x)e^{i\nu_d(x)}\frac{1}{\sqrt{2}}(|A_d(x)\rangle + |B_d(x)\rangle) \quad (26)$$

where the real numbers $m_d(x), n_d(x)$ are such that $0 \leq m_d(x), n_d(x)$ and $m_d(x)^2 + n_d(x)^2 = 1$. The decision measurement on the membership of the exemplar x with respect to the concept ‘ A or B ’ is represented by the orthogonal projection operator $M \oplus (M \otimes \mathbb{1} + \mathbb{1} \otimes M - M \otimes M)$, hence the membership weight of x with respect to ‘ A or B ’ is given by

$$\begin{aligned} \mu_x(A \text{ or } B) &= \langle A \text{ or } B(x)|M \oplus (M \otimes \mathbb{1} + \mathbb{1} \otimes M - M \otimes M)|A \text{ or } B(x)\rangle \\ &= m_d(x)^2(\mu_x(A) + \mu_x(B) - \mu_x(A)\mu_x(B)) + n_d(x)^2\left(\frac{\mu_x(A) + \mu_x(B)}{2} + \Re\langle A_d(x)|M|B_d(x)\rangle\right) \end{aligned}$$

The simplest Fock space that allows the modeling of ‘ A or B ’ is $\mathbb{C}^3 \oplus (\mathbb{C}^3 \otimes \mathbb{C}^3)$. Let us denote by $|1, 0, 0\rangle, |0, 1, 0\rangle, |0, 0, 1\rangle$ the canonical basis of \mathbb{C}^3 . Then, let us set $a_x(A) = 1 - \mu_x(A)$ and $b_x(B) = 1 - \mu_x(B)$ if $\mu_x(A) + \mu_x(B) \leq 1$, $a_x(A) = \mu_x(A)$ and $b_x(B) = \mu_x(B)$ if $\mu_x(A) + \mu_x(B) > 1$. In Aerts (2009a) and Aerts, Gabora & Sozzo (2012) it has been proved that, independently of the value of $\mu_x(A) + \mu_x(B)$, the interference term $\Re\langle A_d(x)|M|B_d(x)\rangle$ is given by

$$\Re\langle A_d(x)|M|B_d(x)\rangle = \sqrt{1 - a_x(A)}\sqrt{1 - b_x(B)}\cos\theta_d(x) \quad (28)$$

where $\theta_d(x)$ is the ‘interference angle’. The unit vectors $|A_d(x)\rangle$ and $|B_d(x)\rangle$ are instead represented in the canonical basis of \mathbb{C}^3 by

$$|A_d(x)\rangle = \left(\sqrt{a_x(A)}, 0, \sqrt{1 - a_x(A)}\right) \quad (29)$$

$$|B_d(x)\rangle = e^{i\theta_d(x)}\left(\sqrt{\frac{(1 - a_x(A))(1 - b_x(B))}{a_x(A)}}, \sqrt{\frac{a_x(A) + b_x(B) - 1}{a_x(A)}}, -\sqrt{1 - b_x(B)}\right) \quad \text{if } a_x(A) \neq 0 \quad (30)$$

$$|B_d(x)\rangle = e^{i\theta_d(x)}(0, 1, 0) \quad \text{if } a_x(A) = 0 \quad (31)$$

and the interference angle satisfies the condition

$$\theta_d(x) = \arccos\left(\frac{\frac{2}{n_d(x)^2}\left(\mu_x(A \text{ or } B) - m_d(x)^2(1 - \mu_x(A) - \mu_x(B) - \mu_x(A)\mu_x(B))\right) - \mu_x(A) - \mu_x(B)}{\sqrt{1 - a_x(A)}\sqrt{1 - b_x(B)}}\right) \quad (32)$$

if $a_x(A), b_x(B) \neq 1$. The angle $\theta_d(x)$ is instead arbitrary if $a_x(A) = 1$ or $b_x(B) = 1$ (Aerts, 2009a; Aerts, Gabora & Sozzo, 2013).

Let us now come to the representation for the conjunction ‘ A and B ’. Here, the decision measurement for the membership weight of the exemplar x with respect to the concept ‘ A and B ’ is represented in the

Fock space $\mathcal{F} = \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H})$ by the orthogonal projection operator $M \oplus (M \otimes M)$, while the membership weight of x with respect to ‘ A and B ’ is given by¹

$$\begin{aligned}\mu_x(A \text{ and } B) &= \langle A \text{ and } B(x) | M \oplus (M \otimes M) | A \text{ and } B(x) \rangle \\ &= m_c(x)^2 \mu_x(A) \mu_x(B) + n_c(x)^2 \left(\frac{\mu_x(A) + \mu_x(B)}{2} + \Re \langle A_c(x) | M | B_c(x) \rangle \right)\end{aligned}\quad (33)$$

where $m_c(x), n_c(x)$ are such that $0 \leq m_c(x), n_c(x)$ and $m_c(x)^2 + n_c(x)^2 = 1$. The unit vector $|A \text{ and } B(x)\rangle$ is given by

$$|A \text{ and } B(x)\rangle = m_c(x) e^{i\lambda_c(x)} |A_c(x)\rangle \otimes |B_c(x)\rangle + n_c(x) e^{i\nu_c(x)} \frac{1}{\sqrt{2}} (|A_c(x)\rangle + |B_c(x)\rangle) \quad (34)$$

Also in this case, it has been proved that the interference term $\Re \langle A_c(x) | M | B_c(x) \rangle$ is given by

$$\Re \langle A_c(x) | M | B_c(x) \rangle = \sqrt{1 - a_x(A)} \sqrt{1 - a_x(B)} \cos \theta_c(x) \quad (35)$$

in the Fock space $\mathbb{C}^3 \oplus (\mathbb{C}^3 \otimes \mathbb{C}^3)$, $\theta_c(x)$ being the interference angle. The concepts $A_c(x)$ and $B_c(x)$ are respectively represented in the canonical basis $|1, 0, 0\rangle, |0, 1, 0\rangle, |0, 0, 1\rangle$ of \mathbb{C}^3 by the unit vectors

$$|A_c(x)\rangle = \left(\sqrt{a_x(A)}, 0, \sqrt{1 - a_x(A)} \right) \quad (36)$$

$$|B_c(x)\rangle = e^{i\theta_c(x)} \left(\sqrt{\frac{(1 - a_x(A))(1 - b_x(B))}{a_x(A)}}, \sqrt{\frac{a_x(A) + b_x(B) - 1}{a_x(A)}}, -\sqrt{1 - b_x(B)} \right) \quad \text{if } a_x(A) \neq 0 \quad (37)$$

$$|B_c(x)\rangle = e^{i\theta_c(x)} (0, 1, 0) \quad \text{if } a_x(A) = 0 \quad (38)$$

The interference angle satisfies the condition

$$\theta_c(x) = \arccos \left(\frac{\frac{2}{n_c(x)^2} \left(\mu_x(A \text{ and } B) - m_c(x)^2 \mu_x(A) \mu_x(B) \right) - \mu_x(A) - \mu_x(B)}{\sqrt{1 - a_x(A)} \sqrt{1 - b_x(B)}} \right) \quad (39)$$

if $a_x(A), b_x(B) \neq 1$. The angle $\theta_c(x)$ is instead arbitrary if $a_x(A) = 1$ or $b_x(B) = 1$ (Aerts, 2009a; Aerts, Gabora & Sozzo, 2013).

Let us finally particularize Equations (33), (34) and (35) to the conjunctions ‘ A and B ’ and ‘ A and not B ’ in Section 2. We have

$$|A \text{ and } B(x)\rangle = m_{AB}(x) e^{i\lambda_{AB}(x)} |A(x)\rangle \otimes |B(x)\rangle + n_{AB}(x) e^{i\nu_{AB}(x)} \frac{1}{\sqrt{2}} (|A(x)\rangle + |B(x)\rangle) \quad (40)$$

$$|A \text{ and not } B(x)\rangle = m_{AB'}(x) e^{i\lambda_{AB'}(x)} |A(x)\rangle \otimes |\text{not } B(x)\rangle + n_{AB'}(x) e^{i\nu_{AB'}(x)} \frac{1}{\sqrt{2}} (|A(x)\rangle + |\text{not } B(x)\rangle) \quad (41)$$

and

$$\begin{aligned}\mu_x(A \text{ and } B) &= m_{AB}(x)^2 \mu_x(A) \mu_x(B) + n_{AB}(x)^2 \left(\frac{\mu_x(A) + \mu_x(B)}{2} + \right. \\ &\quad \left. \sqrt{1 - a_x(A)} \sqrt{1 - b_x(B)} \cos \theta_{AB}(x) \right)\end{aligned}\quad (42)$$

$$\begin{aligned}\mu_x(A \text{ and not } B) &= m_{AB'}(x)^2 \mu_x(A) \mu_x(\text{not } B) + n_{AB'}(x)^2 \left(\frac{\mu_x(A) + \mu_x(\text{not } B)}{2} + \right. \\ &\quad \left. \sqrt{1 - a_x(A)} \sqrt{1 - b_x(\text{not } B)} \cos \theta_{AB'}(x) \right)\end{aligned}\quad (43)$$

¹The membership weight $\mu_x(A \text{ or } B)$ could have been calculated from the membership weight $\mu_x(A \text{ and } B)$ by observing that the probability that a subject decides for the membership of the exemplar x with respect to the concept ‘ A or B ’ is 1 minus the probability of decision against membership of x with respect to the concept ‘ A and B ’.

in the Fock space $\mathbb{C}^3 \oplus (\mathbb{C}^3 \otimes \mathbb{C}^3)$.

We have thus completed our quantum mathematics representation of the concepts A , B , the negation ‘not B ’ and the conjunctions ‘ A and B ’ and ‘ A and not B ’ in Fock space. In the next section we will see how this representation works for the experimental data in Section 2.

6 Representing the empirical data in Fock space

Equations (27) and (33) in Section 5 contain the quantum probabilistic expressions allowing the modeling of a major part of Hampton’s data (1988a,b). Moreover, we have showed in Sozzo (2014) that Equation (33) can also model (Alxatib & Pelletier, 2012)’s data on borderline vagueness. We show in this section that almost all the data collected in our experiments on ‘ A and B ’ and ‘ A and not B ’ can be modeled in the same Fock space framework.

Let us start from the conjunction ‘ A and B ’. Tables 5a, 6a, 7a and 8a report, for each exemplar x , the values of the interference angle $\theta_{AB}(x)$ and the weights $m_{AB}(x)^2$ and $n_{AB}(x)^2$ which satisfy Equation (42), together with the representation of the unit vectors $|A_{AB}(x)\rangle$ and $|B_{AB}(x)\rangle$ in \mathbb{C}^3 satisfying Equations (36), (37) and (38). Let us consider the exemplar *Olive* which was double overextended in Section 2, since it scored a membership weight $\mu_x(A) = 0.53125$ with respect to *Fruits*, $\mu_x(B) = 0.63125$ with respect to *Vegetables*, and $\mu_x(A \text{ and } B) = 0.65$ with respect to *Fruits And Vegetables*. As we can see from Table 8a, *Olive* can be modeled in the Fock space $\mathbb{C}^3 \oplus (\mathbb{C}^3 \otimes \mathbb{C}^3)$ with an interference angle $\theta_{AB}(x) = 60.48^\circ$, a weight $m_{AB}(x)^2 = 0.3$ in sector 2, and a weight $n_{AB}(x)^2 = 0.7$ in sector 1. The concepts *Fruits* and *Vegetables* are represented by the unit vectors $|A_{AB}(x)\rangle = (0.73, 0, 0.68)$, $|B_{AB}(x)\rangle = e^{i60.48^\circ}(0.69, 0.55, -0.61)$ in \mathbb{C}^3 . An exemplar that in Section 2 had a big overextension with respect to *Pets And Farmyard Animals* was *Goldfish*. *Goldfish* scored $\mu_x(A) = 0.925$ with respect to *Pets*, $\mu_x(B) = 0.16875$ with respect to *Farmyard Animals* and $\mu_x(A \text{ and } B) = 0.425$ with respect to *Pets And Farmyard Animals*. It can be modeled in $\mathbb{C}^3 \oplus (\mathbb{C}^3 \otimes \mathbb{C}^3)$ with $\theta_{AB}(x) = 99.22^\circ$, $m_{AB}(x)^2 = 0.23$ and $n_{AB}(x)^2 = 0.77$. The concept *Pets* is represented by $|A_{AB}(x)\rangle = (0.96, 0, 0.27)$, while the concept *Farmyard Animals* is represented by $|B_{AB}(x)\rangle = e^{i99.22^\circ}(0.38, 0.32, -0.91)$ (Table 7a). Another non-classical exemplar was *Parsley* with respect to *Fruits And Vegetables*. In our Fock space representation, it is possible to model $\mu_x(A) = 0.01875$, $\mu_x(B) = 0.78125$ and $\mu_x(A \text{ and } B) = 0.45$ of *Parsley* with $\theta_{AB}(x) = 45.6^\circ$, $m_{AB}(x)^2 = 0.07$ and $n_{AB}(x)^2 = 0.93$. Hence, the decision process of a subject estimating whether *Parsley* belongs to *Fruits*, *Vegetables* and *Fruits And Vegetables* occurs prevalently in sector 1 of the Fock space $\mathbb{C}^3 \oplus (\mathbb{C}^3 \otimes \mathbb{C}^3)$. The concepts *Fruits* and *Vegetables* are represented by $|A_{AB}(x)\rangle = (0.99, 0, 0.14)$ and $|B_{AB}(x)\rangle = e^{i45.6^\circ}(0.18, 0.45, -0.88)$, respectively (Table 8a). But, our quantum-theoretic framework also allows the modeling of ‘classical data’, that is, data that can be represented in a classical Kolmogorovian probability model (Section 3). Indeed, the exemplar *Shelves* had a membership weight of $\mu_x(A) = 0.85$ with respect to *Home Furnishing*, $\mu_x(B) = 0.93125$ with respect to *Furniture*, and $\mu_x(A \text{ and } B) = 0.8375$ with respect to *Home Furnishing And Furniture*. *Shelves* can be represented in $\mathbb{C}^3 \oplus (\mathbb{C}^3 \otimes \mathbb{C}^3)$ with $\theta_{AB}(x) = 101.54^\circ$, $m_{AB}(x)^2 = 0.42$ and $n_{AB}(x)^2 = 0.58$. The concept *Home Furnishing* is represented by $|A_{AB}(x)\rangle = (0.92, 0, 0.39)$ and the concept is represented by $|B_{AB}(x)\rangle = e^{i101.54^\circ}(0.37, 0.96, -0.26)$ with respect to the exemplar *Shelves* (Table 5a).

Let us now come to the modeling of the conjunction ‘ A and not B ’. Tables 5b, 6b, 7b and 8b report, for each exemplar x , the values of the interference angle $\theta_{AB'}(x)$ and the weights $m_{AB'}(x)^2$ and $n_{AB'}(x)^2$ which satisfy Equation (43), together with the representation of the unit vectors $|A_{AB'}(x)\rangle$ and $|\text{not } B_{AB'}(x)\rangle$ in \mathbb{C}^3 satisfying Equations (36), (37) and (38). Let us start from the data that are classically very problematical. The exemplar *Prize Bull* was double overextended with respect to *Pets And Not Farmyard Animals*, since it scored $\mu_x(A) = 0.13125$ with respect to *Pets*, $\mu_x(\text{not } B) = 0.2625$ with respect to *Not Farmyard Animals* and $\mu_x(A \text{ and not } B) = 0.275$ with respect to *Pets And Not Farmyard Animals*. The exemplar *Prize Bull* can be modeled in Fock space with an interference angle $\theta_{AB'}(x) = 45.11^\circ$ and weights $m_{AB'}(x)^2 = 0.18$

for sector 2 of Fock space and $n_{AB'}(x)^2 = 0.82$ for sector 1. The concepts *Pets* and *Not Farmyard Animals* are represented by $|A_{AB'}(x)\rangle = (0.93, 0, 0.36)$ and $|\text{not } B_{AB'}(x)\rangle = e^{i45.11^\circ}(0.2, 0.84, -0.51)$ with respect to the exemplar *Prize Bull* (Table 7b). The exemplar *Shelves* scored a high overextension with respect to *Home Furnishing And Not Furniture*, since it gave $\mu_x(A) = 0.85$, $\mu_x(\text{not } B) = 0.125$ and $\mu_x(A \text{ and not } B) = 0.3875$. It can be modeled in Fock space with an interference angle $\theta_{AB'}(x) = 87.87^\circ$ and weights $m_{AB'}(x)^2 = 0.29$ and $n_{AB'}(x)^2 = 0.71$. The concepts *Home Furnishing* and *Not Furniture* are represented by $|A_{AB'}(x)\rangle = (0.39, 0, 0.92)$ and $|\text{not } B_{AB'}(x)\rangle = e^{i87.87^\circ}(0.84, 0.41, -0.35)$ with respect to the exemplar *Shelves* (Table 5b). A similar pattern can be observed for the exemplar *Doberman Guard Dog* which scored $\mu_x(A) = 0.88125$, $\mu_x(\text{not } B) = 0.26875$ and $\mu_x(A \text{ and not } B) = 0.55$. Our quantum model works for this exemplar with $\theta_{AB'}(x) = 74.87^\circ$ and weights $m_{AB'}(x)^2 = 0.25$ and $n_{AB'}(x)^2 = 0.75$. The concepts *Pets* and *Not Farmyard Animals* are represented by $|A_{AB'}(x)\rangle = (0.94, 0, 0.34)$ and $|\text{not } B_{AB'}(x)\rangle = e^{i74.87^\circ}(0.31, 0.41, -0.86)$ with respect to the exemplar *Doberman Guard Dog* (Table 7b). Also in this case, the ‘classical data’ can be modeled as well. For example, the exemplar *Yam* scored $\mu_x(A) = 0.375$ with respect to *Fruits*, $\mu_x(\text{not } B) = 0.43125$ with respect to *Not Vegetables* and $\mu_x(A \text{ and not } B) = 0.2375$ with respect to *Fruits And Not Vegetables*. *Yam* has an interference angle $\theta_{AB'}(x) = 94.32^\circ$ and weights $m_{AB'}(x) = 0.64$ and $n_{AB'}(x) = 0.36$. This means that the decision process of a subject estimating whether *Yam* belongs to *Fruits*, *Vegetables* and *Fruits And Vegetables* occurs prevalently in sector 2 of the Fock space $\mathbb{C}^3 \oplus (\mathbb{C}^3 \otimes \mathbb{C}^3)$. The concept *Fruits* is represented by $|A_{AB'}(x)\rangle = (0.79, 0, 0.61)$, while *Not Vegetables* is represented by $|\text{not } B_{AB'}(x)\rangle = e^{i94.32^\circ}(0.51, 0.56, -0.66)$ in the Hilbert space \mathbb{C}^3 (Table 8b).

Our analysis above, together with Tables 5-8, allow one to conclude that our quantum-mechanical model in Fock space satisfactorily represents the majority of experimental data collected in our experiments on concepts and their combinations, which are classically problematical, as we have observed in Section 3. Moreover, our quantum-theoretic framework describes the deviations of these data from classical (fuzzy set) logic and probability theory in terms of genuinely quantum effects. Indeed, a quantum probabilistic model is needed for the whole set of data, which entails the presence of ‘contextuality’ (Aerts, 1986). Also, both ‘superposition’ and ‘interference’ are manifestly present between concepts, both in the conjunction ‘*A* and *B*’ and in the conjunction ‘*A* and not *B*’. And quantum field-theoretic notions, i.e. sector, Fock space, tensor product (Section 4) are required to model our data. For what instead concerns ‘emergence’, ‘emergent dynamics’ – which also strongly occurs – deserves a more detailed analysis and it is connected with our explanatory hypothesis we have recently provided to cope with such deviations from classicality in cognitive and decision processes (Aerts, 2009a; Aerts, Gabora & Sozzo, 2013). We have indeed proposed a mechanism that explains the effectiveness of a quantum-theoretic modeling. It is exactly the quantum effect of emergence which comes into play. More precisely, whenever a given subject is asked to estimate whether a given exemplar *x* belongs to the vague concepts *A*, *B*, ‘*A* and *B*’ (‘*A* and not *B*’), two mechanisms act simultaneously and in superposition in the subject’s thought. A ‘quantum logical thought’, which is a probabilistic version of the classical logical reasoning, where the subject considers two copies of the exemplar *x* and estimates whether the first copy belongs to *A* and the second copy of *x* belongs to *B* (‘not *B*’). But also a ‘quantum conceptual thought’ acts, where the subject estimates whether the exemplar *x* belongs to the newly emergent concept ‘*A* and *B*’ (‘*A* and not *B*’). The place whether these superposed processes can be suitably structured is the Fock space. Sector 1 of Fock space hosts the latter process, while sector 2 hosts the former, while the weights $m_{AB}^2(x)$ and $n_{AB}^2(x)$ ($m_{AB'}^2(x)$ and $n_{AB'}^2(x)$) measure the amount of ‘participation’ of sectors 2 and 1, respectively. But, what happens in human thought during a cognitive test is a quantum superposition of both processes. As a consequence of this explanatory hypothesis, an effect, a deviation, or a contradiction, are not failures of classical logical reasoning but, rather, they are a manifestation of the presence of a superposed thought, quantum logical and quantum emergent thought.

It is important to remark, to conclude, that we did not inquire into the relationships between the representation of a concept *A* and that of ‘not *A*’ in the present article. We indeed observe that quantum

logical rules should hold in Sector 2 of Fock space – this was implicitly assumed in the modeling of both the conjunction ‘ A and B ’ and the disjunction ‘ A or B ’ in Section 5. Logical coherence would then lead us to assume that the representation of ‘not A ’ should be constructed from the representation of A by requiring that quantum logical rules – the rules of quantum logical negation, in this case – are valid in Sector 2 of Fock space. We believe that this should be the case, but we also think that a complete analysis of this situation is only possible if data on A , B , ‘not B ’, ‘ A and B ’, ‘ A and not B ’, but also ‘not A ’, ‘not A and B ’ and ‘not A and not B ’ are simultaneously collected. We are presently working on the elaboration of these data and we plan to deal with this interesting aspect in a forthcoming paper (Aerts, Sozzo & Veloz, 2014).

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<i>A=Home Furnishing, B=Furniture</i>						
<i>Exemplar</i>	$\mu_x(A)$	$\mu_x(B)$	$\mu_x(A \text{ and } B)$	$\Delta_{AB}(x)$	$k_{AB}(x)$	$\text{Doub}_{AB}(x)$
<i>Mantelpiece</i>	0.9	0.6125	0.7125	0.1	0.2	0.1875
<i>Window Seat</i>	0.5	0.48125	0.45	-0.03125	0.46875	0.05
<i>Painting</i>	0.8	0.4875	0.6375	0.15	0.35	0.1625
<i>Light Fixture</i>	0.875	0.6	0.725	0.125	0.25	0.15
<i>Kitchen Counter</i>	0.66875	0.4875	0.55	0.0625	0.39375	0.11875
<i>Bath Tub</i>	0.725	0.5125	0.5875	0.075	0.35	0.1375
<i>Deck Chair</i>	0.73125	0.9	0.7375	0.00625	0.10625	0.1625
<i>Shelves</i>	0.85	0.93125	0.8375	-0.0125	0.05625	0.09375
<i>Rug</i>	0.89375	0.575	0.7	0.125	0.23125	0.19375
<i>Bed</i>	0.75625	0.925	0.7875	0.03125	0.10625	0.1375
<i>Wall-Hangings</i>	0.86875	0.4625	0.55	0.0875	0.21875	0.31875
<i>Space Rack</i>	0.375	0.425	0.4125	0.0375	0.6125	0.0125
<i>Ashtray</i>	0.74375	0.4	0.4875	0.0875	0.34375	0.25625
<i>Bar</i>	0.71875	0.625	0.6125	-0.0125	0.26875	0.10625
<i>Lamp</i>	0.94375	0.64375	0.75	0.10625	0.1625	0.19375
<i>Wall Mirror</i>	0.9125	0.75625	0.825	0.06875	0.15625	0.0875
<i>Door Bell</i>	0.75	0.33125	0.5	0.16875	0.41875	0.25
<i>Hammock</i>	0.61875	0.6625	0.6	-0.01875	0.31875	0.0625
<i>Desk</i>	0.78125	0.95	0.775	-0.00625	0.04375	0.175
<i>Refrigerator</i>	0.74375	0.725	0.6625	-0.0625	0.19375	0.08125
<i>Park Bench</i>	0.53125	0.6625	0.55	0.01875	0.35625	0.1125
<i>Waste Paper Basket</i>	0.69375	0.54375	0.5875	0.04375	0.35	0.10625
<i>Sculpture</i>	0.825	0.4625	0.575	0.1125	0.2875	0.25
<i>Sink Unit</i>	0.70625	0.56875	0.6	0.03125	0.325	0.10625

Table 1a. Membership weights with respect to the concepts *Home Furnishing*, *Furniture* and their conjunction *Home Furnishing And Furniture*.

<i>A=Home Furnishing, B=Furniture</i>							
<i>Exemplar</i>	$\mu_x(A)$	$\mu_x(\text{not } B)$	$\mu_x(A \text{ and not } B)$	$\Delta_{AB'}(x)$	$k_{AB'}(x)$	$\text{Doub}_{AB'}(x)$	$l_{BB'}(x)$
<i>Mantelpiece</i>	0.9	0.5	0.75	0.25	0.35	0.15	-0.1125
<i>Window Seat</i>	0.5	0.55	0.4875	-0.0125	0.4375	0.0625	-0.03125
<i>Painting</i>	0.8	0.64375	0.6	-0.04375	0.15625	0.2	-0.13125
<i>Light Fixture</i>	0.875	0.5125	0.625	0.1125	0.2375	0.25	-0.1125
<i>Kitchen Counter</i>	0.66875	0.61875	0.5375	-0.08125	0.25	0.13125	-0.10625
<i>Bath Tub</i>	0.725	0.4625	0.5875	0.125	0.4	0.1375	0.025
<i>Deck Chair</i>	0.73125	0.2	0.4125	0.2125	0.48125	0.31875	-0.1
<i>Shelves</i>	0.85	0.125	0.3875	0.2625	0.4125	0.4625	-0.05625
<i>Rug</i>	0.89375	0.60625	0.675	0.06875	0.175	0.21875	-0.18125
<i>Bed</i>	0.75625	0.10625	0.3625	0.25625	0.5	0.39375	-0.03125
<i>Wall-Hangings</i>	0.86875	0.68125	0.7125	0.03125	0.1625	0.15625	-0.14375
<i>Space Rack</i>	0.375	0.61875	0.4875	0.1125	0.49375	0.13125	-0.04375
<i>Ashtray</i>	0.74375	0.6375	0.6	-0.0375	0.21875	0.14375	-0.0375
<i>Bar</i>	0.71875	0.50625	0.6125	0.10625	0.3875	0.10625	-0.13125
<i>Lamp</i>	0.94375	0.4875	0.7	0.2125	0.26875	0.24375	-0.13125
<i>Wall Mirror</i>	0.9125	0.45	0.6625	0.2125	0.3	0.25	-0.20625
<i>Door Bell</i>	0.75	0.7875	0.6375	-0.1125	0.1	0.15	-0.11875
<i>Hammock</i>	0.61875	0.40625	0.5	0.09375	0.475	0.11875	-0.06875
<i>Desk</i>	0.78125	0.0875	0.325	0.2375	0.45625	0.45625	-0.0375
<i>Refrigerator</i>	0.74375	0.40625	0.55	0.14375	0.4	0.19375	-0.13125
<i>Park Bench</i>	0.53125	0.45625	0.2875	-0.16875	0.3	0.24375	-0.11875
<i>Waste Paper Basket</i>	0.69375	0.63125	0.4125	-0.21875	0.0875	0.28125	-0.175
<i>Sculpture</i>	0.825	0.65625	0.725	0.06875	0.24375	0.1	-0.11875
<i>Sink Unit</i>	0.70625	0.575	0.5625	-0.0125	0.28125	0.14375	-0.14375

Table 1b. Membership weights with respect to the concepts *Home Furnishing*, *Not Furniture* and their conjunction *Home Furnishing And Not Furniture*.

<i>A=Spices, B=Herbs</i>						
<i>Exemplar</i>	$\mu_x(A)$	$\mu_x(B)$	$\mu_x(A \text{ and } B)$	$\Delta_{AB}(x)$	$k_{AB}(x)$	$\text{Doub}_{AB}(x)$
<i>Molasses</i>	0.3625	0.13125	0.2375	0.10625	0.74375	0.125
<i>Salt</i>	0.66875	0.04375	0.2375	0.19375	0.525	0.43125
<i>Peppermint</i>	0.66875	0.925	0.7	0.03125	0.10625	0.225
<i>Curry</i>	0.9625	0.28125	0.5375	0.25625	0.29375	0.425
<i>Oregano</i>	0.8125	0.85625	0.7875	-0.025	0.11875	0.06875
<i>MSG</i>	0.44375	0.11875	0.225	0.10625	0.6625	0.21875
<i>Chili Pepper</i>	0.975	0.53125	0.8	0.26875	0.29375	0.175
<i>Mustard</i>	0.65	0.275	0.4875	0.2125	0.5625	0.1625
<i>Mint</i>	0.64375	0.95625	0.7875	0.14375	0.1875	0.16875
<i>Cinnamon</i>	1	0.49375	0.6875	0.19375	0.19375	0.3125
<i>Parsley</i>	0.5375	0.9	0.675	0.1375	0.2375	0.225
<i>Saccharin</i>	0.34375	0.1375	0.2375	0.1	0.75625	0.10625
<i>Poppy Seeds</i>	0.81875	0.46875	0.5875	0.11875	0.3	0.23125
<i>Pepper</i>	0.99375	0.46875	0.7	0.23125	0.2375	0.29375
<i>Turmeric</i>	0.88125	0.525	0.7375	0.2125	0.33125	0.14375
<i>Sugar</i>	0.45	0.34375	0.35	0.00625	0.55625	0.1
<i>Vinegar</i>	0.3	0.10625	0.15	0.04375	0.74375	0.15
<i>Sesame Seeds</i>	0.8	0.4875	0.5875	0.1	0.3	0.2125
<i>Lemon Juice</i>	0.275	0.2	0.15	-0.05	0.675	0.125
<i>Chocolate</i>	0.26875	0.2125	0.2	-0.0125	0.71875	0.06875
<i>Horseradish</i>	0.6125	0.66875	0.6125	0	0.33125	0.05625
<i>Vanilla</i>	0.7625	0.5125	0.625	0.1125	0.35	0.1375
<i>Chives</i>	0.6625	0.8875	0.7625	0.1	0.2125	0.125
<i>Root Ginger</i>	0.84375	0.5625	0.6875	0.125	0.28125	0.15625

Table 2a. Membership weights with respect to the concepts *Spices*, *Herbs* and their conjunction *Spices And Herbs*.

<i>A=Spices, B=Herbs</i>							
<i>Exemplar</i>	$\mu_x(A)$	$\mu_x(\text{not } B)$	$\mu_x(A \text{ and not } B)$	$\Delta_{AB'}(x)$	$k_{AB'}(x)$	$\text{Doub}_{AB'}(x)$	$l_{BB'}(x)$
<i>Molasses</i>	0.3625	0.8375	0.5375	0.175	0.3375	0.3	0.03125
<i>Salt</i>	0.66875	0.91875	0.6875	0.01875	0.1	0.23125	0.0375
<i>Peppermint</i>	0.66875	0.1	0.375	0.275	0.60625	0.29375	-0.025
<i>Curry</i>	0.9625	0.775	0.875	0.1	0.1375	0.0875	-0.05625
<i>Oregano</i>	0.8125	0.125	0.4	0.275	0.4625	0.4125	0.01875
<i>MSG</i>	0.44375	0.85	0.575	0.13125	0.28125	0.275	0.03125
<i>Chili Pepper</i>	0.975	0.5625	0.9	0.3375	0.3625	0.075	-0.09375
<i>Mustard</i>	0.65	0.70625	0.65	0	0.29375	0.05625	0.01875
<i>Mint</i>	0.64375	0.0875	0.3125	0.225	0.58125	0.33125	-0.04375
<i>Cinnamon</i>	1	0.5125	0.7875	0.275	0.275	0.2125	-0.00625
<i>Parsley</i>	0.5375	0.0875	0.2625	0.175	0.6375	0.275	0.0125
<i>Saccarin</i>	0.34375	0.875	0.5375	0.19375	0.31875	0.3375	-0.0125
<i>Poppy Seeds</i>	0.81875	0.5375	0.6625	0.125	0.30625	0.15625	-0.00625
<i>Pepper</i>	0.99375	0.58125	0.9	0.31875	0.325	0.09375	-0.05
<i>Turmeric</i>	0.88125	0.43125	0.6875	0.25625	0.375	0.19375	0.04375
<i>Sugar</i>	0.45	0.76875	0.5625	0.1125	0.34375	0.20625	-0.1125
<i>Vinegar</i>	0.3	0.88125	0.4125	0.1125	0.23125	0.46875	0.0125
<i>Sesame Seeds</i>	0.8	0.5875	0.7	0.1125	0.3125	0.1	-0.075
<i>Lemon Juice</i>	0.275	0.80625	0.425	0.15	0.34375	0.38125	-0.00625
<i>Chocolate</i>	0.26875	0.8	0.4625	0.19375	0.39375	0.3375	-0.0125
<i>Horseradish</i>	0.6125	0.28125	0.4	0.11875	0.50625	0.2125	0.05
<i>Vanilla</i>	0.7625	0.4875	0.6125	0.125	0.3625	0.15	0
<i>Chives</i>	0.6625	0.25625	0.275	0.01875	0.35625	0.3875	-0.14375
<i>Root Ginger</i>	0.84375	0.44375	0.5875	0.14375	0.3	0.25625	-0.00625

Table 2b. Membership weights with respect to the concepts *Spices*, *Not Herbs* and their conjunction *Spices And Not Herbs*.

<i>A=Pets, B=Farmyard Animals</i>						
<i>Exemplar</i>	$\mu_x(A)$	$\mu_x(B)$	$\mu_x(A \text{ and } B)$	$\Delta_{AB}(x)$	$k_{AB}(x)$	$\text{Doub}_{AB}(x)$
<i>Goldfish</i>	0.925	0.16875	0.425	0.25625	0.33125	0.5
<i>Robin</i>	0.275	0.3625	0.3125	0.0375	0.675	0.05
<i>Blue-tit</i>	0.25	0.3125	0.175	-0.075	0.6125	0.1375
<i>Collie Dog</i>	0.95	0.76875	0.8625	0.09375	0.14375	0.0875
<i>Camel</i>	0.15625	0.25625	0.2	0.04375	0.7875	0.05625
<i>Squirrel</i>	0.3	0.39375	0.275	-0.025	0.58125	0.11875
<i>Guide Dog for Blind</i>	0.925	0.325	0.55	0.225	0.3	0.375
<i>Spider</i>	0.3125	0.3875	0.3125	0	0.6125	0.075
<i>Homing Pigeon</i>	0.40625	0.70625	0.5625	0.15625	0.45	0.14375
<i>Monkey</i>	0.39375	0.175	0.2	0.025	0.63125	0.19375
<i>Circus Horse</i>	0.3	0.48125	0.3375	0.0375	0.55625	0.14375
<i>Prize Bull</i>	0.13125	0.7625	0.425	0.29375	0.53125	0.3375
<i>Rat</i>	0.2	0.35625	0.2125	0.0125	0.65625	0.14375
<i>Badger</i>	0.1625	0.275	0.1375	-0.025	0.7	0.1375
<i>Siamese Cat</i>	0.9875	0.5	0.7375	0.2375	0.25	0.25
<i>Race Horse</i>	0.2875	0.7	0.5125	0.225	0.525	0.1875
<i>Fox</i>	0.13125	0.3	0.175	0.04375	0.74375	0.125
<i>Donkey</i>	0.2875	0.9	0.5625	0.275	0.375	0.3375
<i>Field Mouse</i>	0.1625	0.40625	0.225	0.0625	0.65625	0.18125
<i>Ginger Tom-cat</i>	0.81875	0.50625	0.5875	0.08125	0.2625	0.23125
<i>Husky in Slead Team</i>	0.64375	0.50625	0.5625	0.05625	0.4125	0.08125
<i>Cart Horse</i>	0.26875	0.8625	0.525	0.25625	0.39375	0.3375
<i>Chicken</i>	0.23125	0.95	0.575	0.34375	0.39375	0.375
<i>Doberman Guard Dog</i>	0.88125	0.75625	0.8	0.04375	0.1625	0.08125

Table 3a. Membership weights with respect to the concepts *Pets*, *Farmyard Animals* and their conjunction *Pets And Farmyard Animals*.

<i>A=Pets, B=Farmyard Animals</i>							
<i>Exemplar</i>	$\mu_x(A)$	$\mu_x(\text{not } B)$	$\mu_x(A \text{ and not } B)$	$\Delta_{AB'}(x)$	$k_{AB'}(x)$	$\text{Doub}_{AB'}(x)$	$l_{BB'}(x)$
<i>Goldfish</i>	0.925	0.8125	0.9125	0.1	0.175	0.0125	0.01875
<i>Robin</i>	0.275	0.6375	0.35	0.075	0.4375	0.2875	0
<i>Blue-tit</i>	0.25	0.7125	0.3875	0.1375	0.425	0.325	-0.025
<i>Collie Dog</i>	0.95	0.35	0.5625	0.2125	0.2625	0.3875	-0.11875
<i>Camel</i>	0.15625	0.75	0.3125	0.15625	0.40625	0.4375	-0.00625
<i>Squirrel</i>	0.3	0.65	0.2625	-0.0375	0.3125	0.3875	-0.04375
<i>Guide Dog for Blind</i>	0.925	0.69375	0.725	0.03125	0.10625	0.2	-0.01875
<i>Spider</i>	0.3125	0.63125	0.3125	0	0.36875	0.31875	-0.01875
<i>Homing Pigeon</i>	0.40625	0.3375	0.25	-0.0875	0.50625	0.15625	-0.04375
<i>Monkey</i>	0.39375	0.79375	0.4875	0.09375	0.3	0.30625	0.03125
<i>Circus Horse</i>	0.3	0.6	0.35	0.05	0.45	0.25	-0.08125
<i>Prize Bull</i>	0.13125	0.2625	0.275	0.14375	0.88125	-0.0125	-0.025
<i>Rat</i>	0.2	0.675	0.275	0.075	0.4	0.4	-0.03125
<i>Badger</i>	0.1625	0.73125	0.2625	0.1	0.36875	0.46875	-0.00625
<i>Siamese Cat</i>	0.9875	0.525	0.75	0.225	0.2375	0.2375	-0.025
<i>Race Horse</i>	0.2875	0.3875	0.3125	0.025	0.6375	0.075	-0.0875
<i>Fox</i>	0.13125	0.68125	0.2875	0.15625	0.475	0.39375	0.01875
<i>Donkey</i>	0.2875	0.15	0.175	0.025	0.7375	0.1125	-0.05
<i>Field Mouse</i>	0.1625	0.5875	0.2375	0.075	0.4875	0.35	0.00625
<i>Ginger Tom-cat</i>	0.81875	0.54375	0.575	0.03125	0.2125	0.24375	-0.05
<i>Husky in Slead team</i>	0.64375	0.525	0.5125	-0.0125	0.34375	0.13125	-0.03125
<i>Cart Horse</i>	0.26875	0.15	0.2	0.05	0.78125	0.06875	-0.0125
<i>Chicken</i>	0.23125	0.0625	0.1125	0.05	0.81875	0.11875	-0.0125
<i>Doberman Guard Dog</i>	0.88125	0.26875	0.55	0.28125	0.4	0.33125	-0.025

Table 3b. Membership weights with respect to the concepts *Pets*, *Not Farmyard Animals* and their conjunction *Pets And Not Farmyard Animals*.

<i>A=Fruits, B=Vegetables</i>						
<i>Exemplar</i>	$\mu_x(A)$	$\mu_x(B)$	$\mu_x(A \text{ and } B)$	$\Delta_{AB}(x)$	$k_{AB}(x)$	$\text{Doub}_{AB}(x)$
<i>Apple</i>	1	0.225	0.6	0.375	0.375	0.4
<i>Parsley</i>	0.01875	0.78125	0.45	0.43125	0.65	0.33125
<i>Olive</i>	0.53125	0.63125	0.65	0.11875	0.4875	-0.01875
<i>Chili Pepper</i>	0.1875	0.73125	0.5125	0.325	0.59375	0.21875
<i>Broccoli</i>	0.09375	1	0.5875	0.49375	0.49375	0.4125
<i>Root Ginger</i>	0.1375	0.7125	0.4625	0.325	0.6125	0.25
<i>Pumpkin</i>	0.45	0.775	0.6625	0.2125	0.4375	0.1125
<i>Raisin</i>	0.88125	0.26875	0.525	0.25625	0.375	0.35625
<i>Acorn</i>	0.5875	0.4	0.4625	0.0625	0.475	0.125
<i>Mustard</i>	0.06875	0.3875	0.2875	0.21875	0.83125	0.1
<i>Rice</i>	0.11875	0.45625	0.2125	0.09375	0.6375	0.24375
<i>Tomato</i>	0.3375	0.8875	0.7	0.3625	0.475	0.1875
<i>Coconut</i>	0.925	0.31875	0.5625	0.24375	0.31875	0.3625
<i>Mushroom</i>	0.11875	0.6625	0.325	0.20625	0.54375	0.3375
<i>Wheat</i>	0.16875	0.50625	0.3375	0.16875	0.6625	0.16875
<i>Green Pepper</i>	0.225	0.6125	0.4875	0.2625	0.65	0.125
<i>Watercress</i>	0.1375	0.7625	0.4875	0.35	0.5875	0.275
<i>Peanut</i>	0.61875	0.29375	0.475	0.18125	0.5625	0.14375
<i>Black Pepper</i>	0.20625	0.4125	0.375	0.16875	0.75625	0.0375
<i>Garlic</i>	0.125	0.7875	0.525	0.4	0.6125	0.2625
<i>Yam</i>	0.375	0.65625	0.5875	0.2125	0.55625	0.06875
<i>Elderberry</i>	0.50625	0.39375	0.45	0.05625	0.55	0.05625
<i>Almond</i>	0.7625	0.29375	0.475	0.18125	0.41875	0.2875
<i>Lentils</i>	0.1125	0.6625	0.375	0.2625	0.6	0.2875

Table 4a. Membership weights with respect to the concepts *Fruits*, *Vegetables* and their conjunction *Fruits And Vegetables*.

<i>A=Fruits, B=Vegetables</i>							
<i>Exemplar</i>	$\mu_x(A)$	$\mu_x(\text{not } B)$	$\mu_x(A \text{ and not } B)$	$\Delta_{AB'}(x)$	$k_{AB'}(x)$	$\text{Doub}_{AB'}(x)$	$l_{BB'}(x)$
<i>Apple</i>	1	0.81875	0.8875	0.06875	0.06875	0.1125	-0.04375
<i>Parsley</i>	0.01875	0.25	0.1	0.08125	0.83125	0.15	-0.03125
<i>Olive</i>	0.53125	0.44375	0.3375	-0.10625	0.3625	0.19375	-0.075
<i>Chili Pepper</i>	0.1875	0.35	0.2	0.0125	0.6625	0.15	-0.08125
<i>Broccoli</i>	0.09375	0.0625	0.0875	0.025	0.93125	0.00625	-0.0625
<i>Root Ginger</i>	0.1375	0.325	0.1375	0	0.675	0.1875	-0.0375
<i>Pumpkin</i>	0.45	0.2625	0.2125	-0.05	0.5	0.2375	-0.0375
<i>Raisin</i>	0.88125	0.7625	0.75	-0.0125	0.10625	0.13125	-0.03125
<i>Acorn</i>	0.5875	0.64375	0.4875	-0.1	0.25625	0.15625	-0.04375
<i>Mustard</i>	0.06875	0.6	0.225	0.15625	0.55625	0.375	0.0125
<i>Rice</i>	0.11875	0.51875	0.225	0.10625	0.5875	0.29375	0.025
<i>Tomato</i>	0.3375	0.1875	0.2	0.0125	0.675	0.1375	-0.075
<i>Coconut</i>	0.925	0.7	0.6875	-0.0125	0.0625	0.2375	-0.01875
<i>Mushroom</i>	0.11875	0.38125	0.125	0.00625	0.625	0.25625	-0.04375
<i>Wheat</i>	0.16875	0.51875	0.2125	0.04375	0.525	0.30625	-0.025
<i>Green Pepper</i>	0.225	0.40625	0.2375	0.0125	0.60625	0.16875	-0.01875
<i>Watercress</i>	0.1375	0.25	0.1	-0.0375	0.7125	0.15	-0.0125
<i>Peanut</i>	0.61875	0.75	0.55	-0.06875	0.18125	0.2	-0.04375
<i>Black Pepper</i>	0.20625	0.6125	0.2125	0.00625	0.39375	0.4	-0.025
<i>Garlic</i>	0.125	0.24375	0.1	-0.025	0.73125	0.14375	-0.03125
<i>Yam</i>	0.375	0.43125	0.2375	-0.1375	0.43125	0.19375	-0.0875
<i>Elderberry</i>	0.50625	0.60625	0.4125	-0.09375	0.3	0.19375	0
<i>Almond</i>	0.7625	0.71875	0.6125	-0.10625	0.13125	0.15	-0.0125
<i>Lentils</i>	0.1125	0.375	0.1125	0	0.625	0.2625	-0.0375

Table 4b. Membership weights with respect to the concepts *Fruits*, *Not Vegetables* and their conjunction *Pets And Not Vegetables*.

<i>A=Home Furnishing, B=Furniture</i>								
<i>Exemplar</i>	$\mu_x(A)$	$\mu_x(B)$	$\mu_x(A \text{ and } B)$	$\theta_{AB}(x)$	$m_{AB}(x)^2$	$n_{AB}(x)^2$	$ A_{AB}(x)\rangle$	$e^{-i\theta_{AB}(x)} B_{AB}(x)\rangle$
<i>Mantelpiece</i>	0.9	0.6125	0.7125	82.84	0.3	0.7	(0.95, 0, 0.32)	(0.43, 0.75, -0.62)
<i>Window Seat</i>	0.5	0.48125	0.45	74.75	0.45	0.55	(0.71, 0, 0.71)	(0.76, 0.19, -0.69)
<i>Painting</i>	0.8	0.4875	0.6375	71.12	0.31	0.69	(0.89, 0, 0.45)	(0.52, 0.6, -0.72)
<i>Light Fixture</i>	0.875	0.6	0.725	73.2	0.28	0.72	(0.94, 0, 0.35)	(0.55, 0.74, -0.63)
<i>Kitchen Counter</i>	0.66875	0.4875	0.55	73.91	0.39	0.61	(0.82, 0, 0.58)	(0.72, 0.48, -0.72)
<i>Bath Tub</i>	0.725	0.5125	0.5875	74.9	0.37	0.63	(0.85, 0, 0.52)	(0.68, 0.57, -0.7)
<i>Deck Chair</i>	0.73125	0.9	0.7375	98.46	0.4	0.6	(0.86, 0, 0.52)	(0.39, 0.93, -0.32)
<i>Shelves</i>	0.85	0.93125	0.8375	101.54	0.42	0.58	(0.92, 0, 0.39)	(0.37, 0.96, -0.26)
<i>Rug</i>	0.89375	0.575	0.7	79.31	0.28	0.72	(0.95, 0, 0.33)	(0.46, 0.72, -0.65)
<i>Bed</i>	0.75625	0.925	0.7875	93.14	0.34	0.66	(0.87, 0, 0.49)	(0.36, 0.95, -0.27)
<i>Wall-Hangings</i>	0.86875	0.4625	0.55	95.81	0.37	0.63	(0.93, 0, 0.36)	(0.55, 0.62, -0.73)
<i>Space Rack</i>	0.375	0.425	0.4125	68.21	0.35	0.65	(0.79, 0, 0.61)	(0.53, 0.57, -0.65)
<i>Ashtray</i>	0.74375	0.4	0.4875	82.43	0.42	0.58	(0.86, 0, 0.51)	(0.63, 0.44, -0.77)
<i>Bar</i>	0.71875	0.625	0.6125	80.53	0.41	0.59	(0.85, 0, 0.53)	(0.51, 0.69, -0.61)
<i>Lamp</i>	0.94375	0.64375	0.75	88	0.25	0.75	(0.97, 0, 0.24)	(0.32, 0.79, -0.6)
<i>Wall Mirror</i>	0.9125	0.75625	0.825	73.68	0.27	0.73	(0.96, 0, 0.3)	(0.39, 0.86, -0.49)
<i>Door Bell</i>	0.75	0.33125	0.5	75.7	0.36	0.64	(0.87, 0, 0.5)	(0.57, 0.33, -0.82)
<i>Hammock</i>	0.61875	0.6625	0.6	76.5	0.4	0.6	(0.79, 0, 0.62)	(0.64, 0.67, -0.58)
<i>Desk</i>	0.78125	0.95	0.775	130.06	0.42	0.58	(0.88, 0, 0.47)	(0.27, 0.97, -0.22)
<i>Refrigerator</i>	0.74375	0.725	0.6625	85.71	0.43	0.57	(0.86, 0, 0.51)	(0.53, 0.79, -0.52)
<i>Park Bench</i>	0.53125	0.6625	0.55	76.77	0.41	0.59	(0.73, 0, 0.68)	(0.64, 0.6, -0.58)
<i>Waste Paper Basket</i>	0.69375	0.54375	0.5875	74.8	0.38	0.62	(0.83, 0, 0.55)	(0.54, 0.59, -0.68)
<i>Sculpture</i>	0.825	0.4625	0.575	82.95	0.35	0.65	(0.91, 0, 0.42)	(0.51, 0.59, -0.73)
<i>Sink Unit</i>	0.70625	0.56875	0.6	76.05	0.38	0.62	(0.84, 0, 0.54)	(0.58, 0.62, -0.66)

Table 5a. Representation of A , B and ‘ A and B ’ in the case of the concepts *Home Furnishing* and *Furniture*. Note that the angles are expressed in degrees.

<i>A=Home Furnishing, B=Furniture</i>								
<i>Exemplar</i>	$\mu_x(A)$	$\mu_x(\text{not } B)$	$\mu_x(A \text{ and not } B)$	$\theta_{AB'}(x)$	$m_{AB'}(x)^2$	$n_{AB'}(x)^2$	$ A_{AB'}(x)\rangle$	$e^{-i\theta_{AB'}(x)} \text{not } B_{AB'}(x)\rangle$
<i>Mantelpiece</i>	0.9	0.5	0.75	57.08	0.19	0.81	(0.95, 0, 0.32)	(0.24, 0.67, -0.71)
<i>Window Seat</i>	0.5	0.55	0.4875	74.53	0.44	0.56	(0.71, 0, 0.71)	(0.67, 0.32, -0.67)
<i>Painting</i>	0.8	0.64375	0.6	97.17	0.51	0.49	(0.89, 0, 0.45)	(0.3, 0.74, -0.6)
<i>Light Fixture</i>	0.875	0.5125	0.625	86.17	0.33	0.67	(0.94, 0, 0.35)	(0.26, 0.67, -0.7)
<i>Kitchen Counter</i>	0.66875	0.61875	0.5375	87.4	0.5	0.5	(0.82, 0, 0.58)	(0.43, 0.66, -0.62)
<i>Bath Tub</i>	0.725	0.4625	0.5875	70.93	0.34	0.66	(0.85, 0, 0.52)	(0.45, 0.51, -0.73)
<i>Deck Chair</i>	0.73125	0.2	0.4125	77.99	0.33	0.67	(0.52, 0, 0.86)	(0.74, 0.51, -0.45)
<i>Shelves</i>	0.85	0.125	0.3875	87.87	0.29	0.71	(0.39, 0, 0.92)	(0.84, 0.41, -0.35)
<i>Rug</i>	0.89375	0.60625	0.675	91.51	0.34	0.66	(0.95, 0, 0.33)	(0.22, 0.75, -0.63)
<i>Bed</i>	0.75625	0.10625	0.3625	84.04	0.26	0.74	(0.49, 0, 0.87)	(0.57, 0.75, -0.33)
<i>Wall-Hangings</i>	0.86875	0.68125	0.7125	87.79	0.37	0.63	(0.93, 0, 0.36)	(0.22, 0.8, -0.56)
<i>Space Rack</i>	0.375	0.61875	0.4875	71.12	0.39	0.61	(0.79, 0, 0.61)	(0.61, 0.1, -0.79)
<i>Ashtray</i>	0.74375	0.6375	0.6	87.3	0.45	0.55	(0.86, 0, 0.51)	(0.35, 0.72, -0.6)
<i>Bar</i>	0.71875	0.50625	0.6125	70	0.34	0.66	(0.85, 0, 0.53)	(0.44, 0.56, -0.7)
<i>Lamp</i>	0.94375	0.4875	0.7	75.28	0.2	0.8	(0.97, 0, 0.24)	(0.17, 0.68, -0.72)
<i>Wall Mirror</i>	0.9125	0.45	0.6625	74.9	0.23	0.77	(0.96, 0, 0.3)	(0.23, 0.63, -0.74)
<i>Door Bell</i>	0.75	0.7875	0.6375	104.71	0.61	0.39	(0.87, 0, 0.5)	(0.27, 0.85, -0.46)
<i>Hammock</i>	0.61875	0.40625	0.5	71.51	0.4	0.6	(0.79, 0, 0.62)	(0.6, 0.2, -0.77)
<i>Desk</i>	0.78125	0.0875	0.325	94.73	0.25	0.75	(0.47, 0, 0.88)	(0.56, 0.77, -0.3)
<i>Refrigerator</i>	0.74375	0.40625	0.55	73.67	0.35	0.65	(0.86, 0, 0.51)	(0.45, 0.45, -0.77)
<i>Park Bench</i>	0.53125	0.45625	0.2875	94.77	0.79	0.21	(0.68, 0, 0.73)	(0.72, 0.16, -0.68)
<i>Waste Paper Basket</i>	0.69375	0.63125	0.4125	118.07	0	1	(0.83, 0, 0.55)	(0.4, 0.68, -0.61)
<i>Sculpture</i>	0.825	0.65625	0.725	73.65	0.32	0.68	(0.91, 0, 0.42)	(0.27, 0.76, -0.59)
<i>Sink Unit</i>	0.70625	0.575	0.5625	82.77	0.44	0.56	(0.84, 0, 0.54)	(0.42, 0.63, -0.65)

Table 5b. Representation of A , ‘not B ’ and ‘ A and not B ’ in the case of the concepts *Home Furnishing* and *Furniture*. Note that the angles are expressed in degrees.

<i>A=Spices, B=Herbs</i>								
<i>Exemplar</i>	$\mu_x(A)$	$\mu_x(B)$	$\mu_x(A \text{ and } B)$	$\theta_{AB}(x)$	$m_{AB}(x)^2$	$n_{AB}(x)^2$	$ A_{AB}(x)\rangle$	$e^{-i\theta_{AB}(x)} B_{AB}(x)\rangle$
<i>Molasses</i>	0.3625	0.13125	0.2375	72.46	0.28	0.72	(0.8, 0, 0.6)	(0.26, 0.89, -0.36)
<i>Salt</i>	0.66875	0.04375	0.2375	113.97	0.19	0.81	(0.58, 0, 0.82)	(0.22, 0.93, -0.21)
<i>Peppermint</i>	0.66875	0.925	0.7	107.92	0.37	0.63	(0.82, 0, 0.58)	(0.41, 0.94, -0.27)
<i>Curry</i>	0.9625	0.28125	0.5375	100.93	0.17	0.83	(0.98, 0, 0.19)	(0.36, 0.5, -0.85)
<i>Oregano</i>	0.8125	0.85625	0.7875	86.59	0.38	0.62	(0.9, 0, 0.43)	(0.6, 0.91, -0.38)
<i>MSG</i>	0.44375	0.11875	0.225	84.18	0.32	0.68	(0.75, 0, 0.67)	(0.27, 0.89, -0.34)
<i>Chili Pepper</i>	0.975	0.53125	0.8	44.34	0.1	0.9	(0.99, 0, 0.16)	(0.31, 0.72, -0.68)
<i>Mustard</i>	0.65	0.275	0.4875	66.61	0.32	0.68	(0.59, 0, 0.81)	(0.62, 0.46, -0.52)
<i>Mint</i>	0.64375	0.95625	0.7875	75.75	0.2	0.8	(0.8, 0, 0.6)	(0.37, 0.97, -0.21)
<i>Cinnamon</i>	1	0.49375	0.6875	0	0.23	0.77	(1, 0, 0)	(0, 0.7, -0.71)
<i>Parsley</i>	0.5375	0.9	0.675	81.74	0.28	0.72	(0.73, 0, 0.68)	(0.51, 0.9, -0.32)
<i>Saccharin</i>	0.34375	0.1375	0.2375	70.82	0.28	0.72	(0.81, 0, 0.59)	(0.24, 0.89, -0.37)
<i>Poppy Seeds</i>	0.81875	0.46875	0.5875	80.44	0.35	0.65	(0.9, 0, 0.43)	(0.59, 0.59, -0.73)
<i>Pepper</i>	0.99375	0.46875	0.7	102.84	0.07	0.93	(1, 0, 0.08)	(0.16, 0.68, -0.73)
<i>Turmeric</i>	0.88125	0.525	0.7375	61.68	0.22	0.78	(0.94, 0, 0.34)	(0.52, 0.68, -0.69)
<i>Sugar</i>	0.45	0.34375	0.35	76.84	0.41	0.59	(0.74, 0, 0.67)	(0.49, 0.61, -0.59)
<i>Vinegar</i>	0.3	0.10625	0.15	87.31	0.34	0.66	(0.84, 0, 0.55)	(0.2, 0.92, -0.33)
<i>Sesame Seeds</i>	0.8	0.4875	0.5875	80.12	0.36	0.64	(0.89, 0, 0.45)	(0.6, 0.6, -0.72)
<i>Lemon Juice</i>	0.275	0.2	0.15	91.91	0.46	0.54	(0.85, 0, 0.52)	(0.26, 0.85, -0.45)
<i>Chocolate</i>	0.26875	0.2125	0.2	79.79	0.37	0.63	(0.86, 0, 0.52)	(0.27, 0.84, -0.46)
<i>Horseradish</i>	0.6125	0.66875	0.6125	74.5	0.38	0.62	(0.78, 0, 0.62)	(0.63, 0.68, -0.58)
<i>Vanilla</i>	0.7625	0.5125	0.625	72.11	0.33	0.67	(0.87, 0, 0.49)	(0.58, 0.6, -0.7)
<i>Chives</i>	0.6625	0.8875	0.7625	73.68	0.28	0.72	(0.81, 0, 0.58)	(0.35, 0.91, -0.34)
<i>Root Ginger</i>	0.84375	0.5625	0.6875	73.43	0.3	0.7	(0.92, 0, 0.4)	(0.55, 0.69, -0.66)

Table 6a. Representation of A , B and ‘ A and B ’ in the case of the concepts *Spices* and *Herbs*. Note that the angles are expressed in degrees.

<i>A=Spices, B=Herbs</i>								
<i>Exemplar</i>	$\mu_x(A)$	$\mu_x(\text{not } B)$	$\mu_x(A \text{ and not } B)$	$\theta_{AB'}(x)$	$m_{AB'}(x)^2$	$n_{AB'}(x)^2$	$ A_{AB'}(x)\rangle$	$e^{-i\theta_{AB'}(x)} \text{not } B_{AB'}(x)\rangle$
<i>Molasses</i>	0.3625	0.8375	0.5375	81.2	0.32	0.68	(0.6, 0, 0.8)	(0.53, 0.74, -0.4)
<i>Salt</i>	0.66875	0.91875	0.6875	110.36	0.4	0.6	(0.82, 0, 0.58)	(0.2, 0.94, -0.29)
<i>Peppermint</i>	0.66875	0.1	0.375	72.08	0.22	0.78	(0.58, 0, 0.82)	(0.45, 0.84, -0.32)
<i>Curry</i>	0.9625	0.775	0.875	66.1	0.19	0.81	(0.98, 0, 0.19)	(0.09, 0.88, -0.47)
<i>Oregano</i>	0.8125	0.125	0.4	82.46	0.27	0.73	(0.43, 0, 0.9)	(0.74, 0.58, -0.35)
<i>MSG</i>	0.44375	0.85	0.575	84.41	0.34	0.66	(0.67, 0, 0.75)	(0.43, 0.81, -0.39)
<i>Chili Pepper</i>	0.975	0.5625	0.9	0	0	1.03	(0.99, 0, 0.16)	(0.11, 0.74, -0.66)
<i>Mustard</i>	0.65	0.70625	0.65	75.03	0.37	0.63	(0.81, 0, 0.59)	(0.4, 0.74, -0.54)
<i>Mint</i>	0.64375	0.0875	0.3125	82.93	0.24	0.76	(0.6, 0, 0.8)	(0.4, 0.87, -0.3)
<i>Cinnamon</i>	1	0.5125	0.7875	8.65	0	1	(1, 0, 0)	(0, 0.72, -0.7)
<i>Parsley</i>	0.5375	0.0875	0.2625	83.33	0.26	0.74	(0.68, 0, 0.73)	(0.32, 0.9, -0.3)
<i>Saccharin</i>	0.34375	0.875	0.5375	84.53	0.3	0.7	(0.59, 0, 0.81)	(0.49, 0.8, -0.35)
<i>Poppy Seeds</i>	0.81875	0.5375	0.6625	73.09	0.31	0.69	(0.9, 0, 0.43)	(0.32, 0.66, -0.68)
<i>Pepper</i>	0.99375	0.58125	0.9	0	0	1	(1, 0, 0.08)	(0.05, 0.76, -0.65)
<i>Turmeric</i>	0.88125	0.43125	0.6875	63.09	0.22	0.78	(0.94, 0, 0.34)	(0.28, 0.6, -0.75)
<i>Sugar</i>	0.45	0.76875	0.5625	77.55	0.36	0.64	(0.67, 0, 0.74)	(0.53, 0.7, -0.48)
<i>Vinegar</i>	0.3	0.88125	0.4125	108.16	0.37	0.63	(0.55, 0, 0.84)	(0.53, 0.78, -0.34)
<i>Sesame Seeds</i>	0.8	0.5875	0.7	68.75	0.3	0.7	(0.89, 0, 0.45)	(0.32, 0.7, -0.64)
<i>Lemon Juice</i>	0.275	0.80625	0.425	87.97	0.39	0.61	(0.52, 0, 0.85)	(0.71, 0.54, -0.44)
<i>Chocolate</i>	0.26875	0.8	0.4625	80.83	0.35	0.65	(0.52, 0, 0.86)	(0.74, 0.51, -0.45)
<i>Horseradish</i>	0.6125	0.28125	0.4	76.48	0.39	0.61	(0.62, 0, 0.78)	(0.67, 0.52, -0.53)
<i>Vanilla</i>	0.7625	0.4875	0.6125	72.05	0.33	0.67	(0.87, 0, 0.49)	(0.4, 0.57, -0.72)
<i>Chives</i>	0.6625	0.25625	0.275	96.58	0.57	0.43	(0.58, 0, 0.81)	(0.71, 0.49, -0.51)
<i>Root Ginger</i>	0.84375	0.44375	0.5875	81	0.32	0.68	(0.92, 0, 0.4)	(0.32, 0.58, -0.75)

Table 6b. Representation of A , ‘not B ’ and ‘ A and not B ’ in the case of the concepts *Spices* and *Herbs*. Note that the angles are expressed in degrees.

<i>A=Pets, B=Farmyard Animals</i>								
<i>Exemplar</i>	$\mu_x(A)$	$\mu_x(B)$	$\mu_x(A \text{ and } B)$	$\theta_{AB}(x)$	$m_{AB}(x)^2$	$n_{AB}(x)^2$	$ A_{AB}(x)\rangle$	$e^{-i\theta_{AB}(x)} B_{AB}(x)\rangle$
<i>Goldfish</i>	0.925	0.16875	0.425	99.22	0.23	0.77	(0.96, 0, 0.27)	(0.38, 0.32, -0.91)
<i>Robin</i>	0.275	0.3625	0.3125	71.13	0.34	0.66	(0.85, 0, 0.52)	(0.46, 0.71, -0.6)
<i>Blue-tit</i>	0.25	0.3125	0.175	92.34	0.49	0.51	(0.87, 0, 0.5)	(0.37, 0.76, -0.56)
<i>Collie Dog</i>	0.95	0.76875	0.8625	68.33	0.22	0.78	(0.97, 0, 0.22)	(0.32, 0.87, -0.48)
<i>Camel</i>	0.15625	0.25625	0.2	71.79	0.3	0.7	(0.92, 0, 0.4)	(0.24, 0.83, -0.51)
<i>Squirrel</i>	0.3	0.39375	0.275	82.07	0.43	0.57	(0.84, 0, 0.55)	(0.45, 0.66, -0.63)
<i>Guide Dog for Blind</i>	0.925	0.325	0.55	89.47	0.24	0.76	(0.96, 0, 0.27)	(0.39, 0.52, -0.82)
<i>Spider</i>	0.3125	0.3875	0.3125	76.19	0.39	0.61	(0.83, 0, 0.56)	(0.49, 0.66, -0.62)
<i>Homing Pigeon</i>	0.40625	0.70625	0.5625	69.14	0.34	0.66	(0.64, 0, 0.77)	(0.72, 0.53, -0.54)
<i>Monkey</i>	0.39375	0.175	0.2	88.75	0.41	0.59	(0.78, 0, 0.63)	(0.34, 0.84, -0.42)
<i>Circus Horse</i>	0.3	0.48125	0.3375	78.04	0.41	0.59	(0.84, 0, 0.55)	(0.55, 0.56, -0.69)
<i>Prize Bull</i>	0.13125	0.7625	0.425	73.97	0.25	0.75	(0.93, 0, 0.36)	(0.54, 0.35, -0.87)
<i>Rat</i>	0.2	0.35625	0.2125	84.23	0.4	0.6	(0.89, 0, 0.45)	(0.34, 0.74, -0.6)
<i>Badger</i>	0.1625	0.275	0.1375	92.6	0.44	0.56	(0.92, 0, 0.4)	(0.26, 0.82, -0.52)
<i>Siamese Cat</i>	0.9875	0.5	0.7375	74.53	0.1	0.9	(0.99, 0, 0.11)	(0.16, 0.7, -0.71)
<i>Race Horse</i>	0.2875	0.7	0.5125	67.6	0.33	0.67	(0.84, 0, 0.54)	(0.8, 0.13, -0.84)
<i>Fox</i>	0.13125	0.3	0.175	81.81	0.34	0.66	(0.93, 0, 0.36)	(0.26, 0.81, -0.55)
<i>Donkey</i>	0.2875	0.9	0.5625	76.72	0.23	0.77	(0.54, 0, 0.84)	(0.56, 0.81, -0.32)
<i>Field Mouse</i>	0.1625	0.40625	0.225	83.36	0.36	0.64	(0.92, 0, 0.4)	(0.34, 0.72, -0.64)
<i>Ginger Tom-cat</i>	0.81875	0.50625	0.5875	84.52	0.37	0.63	(0.9, 0, 0.43)	(0.56, 0.63, -0.7)
<i>Husky in Sled team</i>	0.64375	0.50625	0.5625	71.71	0.38	0.62	(0.8, 0, 0.6)	(0.78, 0.48, -0.7)
<i>Cart Horse</i>	0.26875	0.8625	0.525	77.36	0.27	0.73	(0.52, 0, 0.86)	(0.67, 0.7, -0.37)
<i>Chicken</i>	0.23125	0.95	0.575	74.57	0.16	0.84	(0.48, 0, 0.88)	(0.47, 0.89, -0.22)
<i>Doberman Guard Dog</i>	0.88125	0.75625	0.8	76.33	0.31	0.69	(0.94, 0, 0.34)	(0.36, 0.85, -0.49)

Table 7a. Representation of A , B and ‘ A and B ’ in the case of the concepts *Pets* and *Farmyard Animals*. Note that the angles are expressed in degrees.

<i>A=Pets, B=Farmyard Animals</i>								
<i>Exemplar</i>	$\mu_x(A)$	$\mu_x(B)$	$\mu_x(A \text{ and } B)$	$\theta_{AB}(x)$	$m_{AB}(x)^2$	$n_{AB}(x)^2$	$ A_{AB}(x)\rangle$	$e^{-i\theta_{AB}(x)} B_{AB}(x)\rangle$
<i>Goldfish</i>	0.925	0.8125	0.9125	48.35	0.18	0.82	(0.96, 0, 0.27)	(0.12, 0.89, -0.43)
<i>Robin</i>	0.275	0.6375	0.35	84.7	0.45	0.55	(0.85, 0, 0.52)	(0.49, 0.35, -0.8)
<i>Blue-tit</i>	0.25	0.7125	0.3875	82.83	0.41	0.59	(0.87, 0, 0.5)	(0.49, 0.22, -0.84)
<i>Collie Dog</i>	0.95	0.35	0.5625	99.04	0.2	0.8	(0.97, 0, 0.22)	(0.18, 0.56, -0.81)
<i>Camel</i>	0.15625	0.75	0.3125	94.25	0.37	0.63	(0.92, 0, 0.4)	(0.37, 0.33, -0.87)
<i>Squirrel</i>	0.3	0.65	0.2625	98.76	0.68	0.32	(0.84, 0, 0.55)	(0.53, 0.27, -0.81)
<i>Guide Dog for Blind</i>	0.925	0.69375	0.725	103.83	0.37	0.63	(0.96, 0, 0.27)	(0.16, 0.82, -0.55)
<i>Spider</i>	0.3125	0.63125	0.3125	91.03	0.57	0.43	(0.83, 0, 0.56)	(0.54, 0.29, -0.79)
<i>Homing Pigeon</i>	0.40625	0.3375	0.25	89.22	0.53	0.47	(0.77, 0, 0.64)	(0.48, 0.66, -0.58)
<i>Monkey</i>	0.39375	0.79375	0.4875	87.49	0.41	0.59	(0.63, 0, 0.78)	(0.56, 0.69, -0.45)
<i>Circus Horse</i>	0.3	0.6	0.35	83.63	0.46	0.54	(0.84, 0, 0.55)	(0.51, 0.38, -0.77)
<i>Prize Bull</i>	0.13125	0.2625	0.275	45.11	0.18	0.82	(0.93, 0, 0.36)	(0.2, 0.84, -0.51)
<i>Rat</i>	0.2	0.675	0.275	96.25	0.47	0.53	(0.89, 0, 0.45)	(0.41, 0.4, -0.82)
<i>Badger</i>	0.1625	0.73125	0.2625	102.33	0.44	0.56	(0.92, 0, 0.4)	(0.38, 0.36, -0.86)
<i>Siamese Cat</i>	0.9875	0.525	0.75	74.65	0.1	0.9	(0.99, 0, 0.11)	(0.08, 0.72, -0.69)
<i>Race Horse</i>	0.2875	0.3875	0.3125	74.3	0.36	0.64	(0.84, 0, 0.54)	(0.4, 0.68, -0.62)
<i>Fox</i>	0.13125	0.68125	0.2875	93.4	0.34	0.66	(0.93, 0, 0.36)	(0.32, 0.46, -0.83)
<i>Donkey</i>	0.2875	0.15	0.175	82.16	0.35	0.65	(0.84, 0, 0.54)	(0.25, 0.89, -0.39)
<i>Field Mouse</i>	0.1625	0.5875	0.2375	96.42	0.42	0.58	(0.92, 0, 0.4)	(0.34, 0.55, -0.77)
<i>Ginger Tom-cat</i>	0.81875	0.54375	0.575	91.68	0.43	0.57	(0.9, 0, 0.43)	(0.32, 0.67, -0.68)
<i>Husky in Sled Team</i>	0.64375	0.525	0.5125	80.06	0.45	0.55	(0.8, 0, 0.6)	(0.51, 0.51, -0.69)
<i>Cart Horse</i>	0.26875	0.15	0.2	72.68	0.3	0.7	(0.86, 0, 0.52)	(0.23, 0.89, -0.39)
<i>Chicken</i>	0.23125	0.0625	0.1125	86.61	0.3	0.7	(0.88, 0, 0.48)	(0.14, 0.96, -0.25)
<i>Doberman Guard Dog</i>	0.88125	0.26875	0.55	74.87	0.25	0.75	(0.94, 0, 0.34)	(0.31, 0.41, -0.86)

Table 7b. Representation of A , ‘not B ’ and ‘ A and not B ’ in the case of the concepts *Pets* and *Farmyard Animals*. Note that the angles are expressed in degrees.

<i>A=Fruits, B=Vegetables</i>								
<i>Exemplar</i>	$\mu_x(A)$	$\mu_x(B)$	$\mu_x(A \text{ and } B)$	$\theta_{AB}(x)$	$m_{AB}(x)^2$	$n_{AB}(x)^2$	$ A_{AB}(x)\rangle$	$e^{-i\theta_{AB}(x)} B_{AB}(x)\rangle$
<i>Apple</i>	1	0.225	0.6	0	0.03	0.97	(1, 0, 0)	(0, 0.47, -0.88)
<i>Parsley</i>	0.01875	0.78125	0.45	45.6	0.07	0.93	(0.99, 0, 0.14)	(0.18, 0.45, -0.88)
<i>Olive</i>	0.53125	0.63125	0.65	60.48	0.3	0.7	(0.73, 0, 0.68)	(0.69, 0.55, -0.61)
<i>Chili Pepper</i>	0.1875	0.73125	0.5125	61.75	0.25	0.75	(0.9, 0, 0.43)	(0.56, 0.32, -0.86)
<i>Broccoli</i>	0.09375	1	0.5875	0	-0.09	1.09	(0.31, 0, 0.95)	(0, 1, 0)
<i>Root Ginger</i>	0.1375	0.7125	0.4625	63.12	0.22	0.78	(0.93, 0, 0.37)	(0.48, 0.42, -0.84)
<i>Pumpkin</i>	0.45	0.775	0.6625	61.83	0.27	0.73	(0.67, 0, 0.74)	(0.84, 0.71, -0.47)
<i>Raisin</i>	0.88125	0.26875	0.525	79.77	0.26	0.74	(0.94, 0, 0.34)	(0.51, 0.41, -0.86)
<i>Acorn</i>	0.5875	0.4	0.4625	73.7	0.42	0.58	(0.64, 0, 0.77)	(0.68, 0.17, -0.63)
<i>Mustard</i>	0.06875	0.3875	0.2875	48.67	0.16	0.84	(0.97, 0, 0.26)	(0.19, 0.76, -0.62)
<i>Rice</i>	0.11875	0.45625	0.2125	88.8	0.34	0.66	(0.94, 0, 0.34)	(0.3, 0.69, -0.68)
<i>Tomato</i>	0.3375	0.8875	0.7	51.31	0.17	0.83	(0.58, 0, 0.81)	(0.58, 0.82, -0.34)
<i>Coconut</i>	0.925	0.31875	0.5625	85.23	0.23	0.77	(0.96, 0, 0.27)	(0.39, 0.51, -0.83)
<i>Mushroom</i>	0.11875	0.6625	0.325	83.53	0.28	0.72	(0.94, 0, 0.34)	(0.4, 0.5, -0.81)
<i>Wheat</i>	0.16875	0.50625	0.3375	70	0.28	0.72	(0.91, 0, 0.41)	(0.39, 0.63, -0.71)
<i>Green Pepper</i>	0.225	0.6125	0.4875	59.33	0.26	0.74	(0.88, 0, 0.47)	(0.57, 0.46, -0.78)
<i>Watercress</i>	0.1375	0.7625	0.4875	63.35	0.22	0.78	(0.93, 0, 0.37)	(0.55, 0.34, -0.87)
<i>Peanut</i>	0.61875	0.29375	0.475	67.48	0.33	0.67	(0.62, 0, 0.79)	(0.59, 0.48, -0.54)
<i>Black Pepper</i>	0.20625	0.4125	0.375	57	0.24	0.76	(0.89, 0, 0.45)	(0.37, 0.69, -0.64)
<i>Garlic</i>	0.125	0.7875	0.525	57.34	0.19	0.81	(0.94, 0, 0.35)	(0.47, 0.32, -0.89)
<i>Yam</i>	0.375	0.65625	0.5875	61.08	0.31	0.69	(0.61, 0, 0.79)	(0.7, 0.29, -0.59)
<i>Elderberry</i>	0.50625	0.39375	0.45	70	0.38	0.62	(0.7, 0, 0.71)	(0.65, 0.45, -0.63)
<i>Almond</i>	0.7625	0.29375	0.475	77.45	0.36	0.64	(0.87, 0, 0.49)	(0.59, 0.27, -0.84)
<i>Lentils</i>	0.1125	0.6625	0.375	72.62	0.24	0.76	(0.94, 0, 0.34)	(0.38, 0.5, -0.81)

Table 8a. Representation of A , B and ‘ A and B ’ in the case of the concepts *Fruits* and *Vegetables*. Note that the angles are expressed in degrees.

<i>A=Fruits, B=Vegetables</i>								
<i>Exemplar</i>	$\mu_x(A)$	$\mu_x(\text{not } B)$	$\mu_x(A \text{ and not } B)$	$\theta_{AB'}(x)$	$m_{AB'}(x)^2$	$n_{AB'}(x)^2$	$ A_{AB'}(x)\rangle$	$e^{-i\theta_{AB'}(x)} \text{not } B_{AB'}(x)\rangle$
<i>Apple</i>	1	0.81875	0.8875	8.65	0.24	0.76	(1, 0, 0)	(0, 0.9, -0.43)
<i>Parsley</i>	0.01875	0.25	0.1	100.14	0.19	0.81	(0.99, 0, 0.14)	(0.07, 0.86, -0.5)
<i>Olive</i>	0.53125	0.44375	0.3375	88	0.62	0.38	(0.68, 0, 0.73)	(0.71, 0.23, -0.67)
<i>Chili Pepper</i>	0.1875	0.35	0.2	85.57	0.4	0.6	(0.9, 0, 0.43)	(0.28, 0.75, -0.59)
<i>Broccoli</i>	0.09375	0.0625	0.0875	62.97	0.24	0.76	(0.95, 0, 0.31)	(0.08, 0.96, -0.25)
<i>Root Ginger</i>	0.1375	0.325	0.1375	96.33	0.43	0.57	(0.93, 0, 0.37)	(0.23, 0.79, -0.57)
<i>Pumpkin</i>	0.45	0.2625	0.2125	94.72	0.55	0.45	(0.74, 0, 0.67)	(0.46, 0.72, -0.51)
<i>Raisin</i>	0.88125	0.7625	0.75	95.34	0.42	0.58	(0.94, 0, 0.34)	(0.18, 0.85, -0.49)
<i>Acorn</i>	0.5875	0.64375	0.4875	89.53	0.55	0.45	(0.77, 0, 0.64)	(0.5, 0.63, -0.6)
<i>Mustard</i>	0.06875	0.6	0.225	102.58	0.26	0.74	(0.97, 0, 0.26)	(0.21, 0.6, -0.77)
<i>Rice</i>	0.11875	0.51875	0.225	92.19	0.34	0.66	(0.94, 0, 0.34)	(0.26, 0.64, -0.72)
<i>Tomato</i>	0.3375	0.1875	0.2	84.39	0.39	0.61	(0.81, 0, 0.58)	(0.31, 0.85, -0.43)
<i>Coconut</i>	0.925	0.7	0.6875	126.44	0.47	0.53	(0.96, 0, 0.27)	(0.16, 0.82, -0.55)
<i>Mushroom</i>	0.11875	0.38125	0.125	105.98	0.45	0.55	(0.94, 0, 0.34)	(0.23, 0.75, -0.62)
<i>Wheat</i>	0.16875	0.51875	0.2125	96.33	0.44	0.56	(0.91, 0, 0.41)	(0.32, 0.61, -0.72)
<i>Green Pepper</i>	0.225	0.40625	0.2375	84.98	0.42	0.58	(0.88, 0, 0.47)	(0.34, 0.69, -0.64)
<i>Watercress</i>	0.1375	0.25	0.1	100.37	0.48	0.52	(0.93, 0, 0.37)	(0.2, 0.84, -0.5)
<i>Peanut</i>	0.61875	0.75	0.55	95.8	0.55	0.45	(0.79, 0, 0.62)	(0.39, 0.77, -0.5)
<i>Black Pepper</i>	0.20625	0.6125	0.2125	103.64	0.57	0.43	(0.89, 0, 0.45)	(0.4, 0.48, -0.78)
<i>Garlic</i>	0.125	0.24375	0.1	98.91	0.45	0.55	(0.94, 0, 0.35)	(0.19, 0.85, -0.49)
<i>Yam</i>	0.375	0.43125	0.2375	94.32	0.64	0.36	(0.79, 0, 0.61)	(0.51, 0.56, -0.66)
<i>Elderberry</i>	0.50625	0.60625	0.4125	89.03	0.59	0.41	(0.71, 0, 0.7)	(0.62, 0.47, -0.63)
<i>Almond</i>	0.7625	0.71875	0.6125	99.72	0.57	0.43	(0.87, 0, 0.49)	(0.3, 0.79, -0.53)
<i>Lentils</i>	0.1125	0.375	0.1125	109.72	0.47	0.53	(0.94, 0, 0.34)	(0.22, 0.76, -0.61)

Table 8b. Representation of A , ‘not B ’ and ‘ A and not B ’ in the case of the concepts *Fruits* and *Vegetables*. Note that the angles are expressed in degrees.